Approximation Algorithms for Capacitated Perishable Inventory Systems with Positive Lead Times

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Managing perishable inventory systems with positive lead times and finite ordering capacities is important but notoriously difficult in both theory and computation. The optimal control policy is extremely complicated, and no effective heuristic policy has been proposed in the literature. In this paper, we develop an easy-to-compute approximation algorithm for this class of problems and prove that it admits a theoretical worst-case performance guarantee under independent and many commonly used positively correlated demand processes. Our worst-case analysis departs significantly from those in the previous studies, requiring several novel ideas. In particular, we introduce a *transient unit-matching rule* to dynamically match the supply and demand units, and the notion of *associated demand processes* provides the right future demand information to establish the desired results. Our numerical study demonstrates the effectiveness of the proposed algorithm.

Keywords: approximation algorithm; perishable inventory; finite capacity; positive lead time; correlated demand; worst-case performance guarantee

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1. Introduction

Perishable products, or products with finite lifetime, are ubiquitous and indispensable to our society. Perishable products such as meat, fruit, vegetable, dairy products, and frozen foods constitute the majority of supermarket sales. They also include virtually all pharmaceuticals (e.g., drugs and vitamins), which represent a multi-billion dollar industry. Blood blank is another salient example of perishables. For instance, whole blood has a legal lifetime of 21 days, after which it must be discarded due to buildup of contaminants (see Nahmias (2011)).

There is a rich body of literature on perishable inventory systems with zero ordering lead time and infinite ordering capacity. However, practical systems usually have finite ordering capacities, and replenishing inventory typically requires certain time to deliver, i.e., the ordering lead time is positive. There are only a few papers in the existing perishable inventory literature that consider positive lead times. Karaesmen et al. (2011) noted that "*there is very little work on extending discrete review heuristics for the zero lead time model to the case of positive lead times*", and they further suggested that *"development of such positive lead time heuristics, no matter what their genesis, would prove valuable both practically and theoretically, making this a potentially attractive avenue for future research"*. Turning to the issue of finite capacity, Karaesmen et al. (2011) wrote "*A significant majority of the research on inventory management or distribution of perishable goods disregards capacity constraints. Models with limited capacity are better representative of the challenges in practice and require innovative heuristic policies."* When the perishable inventory system has both positive lead times and capacity constraints, we are not aware of any study in the literature.

In this paper, we develop the first approximation algorithm to compute the ordering quantities for capacitated systems with positive lead times where the first-in-first-out (FIFO) issuing policy is adopted (see Nahmias (2011) and Karaesmen et al. (2011)). Our proposed algorithm, called the *proportional-balancing policy* (PB for short), is easy to compute and admits a theoretical worst-case performance guarantee. Moreover, our numerical study shows that it performs consistently well.

The algorithmic design of the PB policy is built upon ideas from Chao et al. (2015), who studied a perishable inventory system with infinite capacity and zero lead times, and Levi et al. (2008b), who studied a classical capacitated non-perishable inventory system. The key methodological contribution of this paper lies in the worst-case analysis of the PB policy that significantly departs from the above two studies and requires several new ideas. We summarize the two most important ideas below.

(1) Transient unit-matching mechanism. The conventional approach of establishing worst-case performance guarantees for stochastic inventory systems is to partition the periods (by comparing inventories between two policies) and then carry out the "right" cost amortization of the proposed policy against the optimal policy (OPT for short). When analyzing perishable inventory systems with both positive lead time and ordering capacity, the choice of period partition and cost amortization become tricky and nontrivial. Indeed, for the classical perishable inventory system studied in Chao et al. (2015), the partition of periods, namely, $\mathscr{T}_H$ and $\mathscr{T}_\Pi$, depends only on the aggregate inventory levels. When positive lead time is introduced into the system, the partition of periods depends on both the age of inventory and the pipeline inventory vector (see (16) and (17)). When the ordering capacity also exists, the partition of periods becomes even more complex (see (27) and (28)). In fact, the algebraic approach developed by Chao et al. (2015) for systems with zero lead time and no capacity constraint becomes too loose as the definitions of $\mathscr{T}_H$ are progressively weaker and it cannot be extended to analyze our system (see §4.2).

To overcome the above difficulty, we develop a novel *transient unit-matching mechanism* to dynamically match the supply and demand units (Proposition 6). The key idea is to decompose the inventory units into two sets – the set of *effective* units that will eventually meet demands, and the set of *perishing* units that will eventually expire. For the former set, we establish permanent pairs between units of PB and OPT (that are used to satisfy the same demand unit), which allow us to compare their associated costs. For the latter set, a permanent one-to-one matching cannot be established; we set up temporary pairs and show that each unit of OPT may be paired multiple but bounded times with different units of PB. Different from the conventional permanent unit-matching techniques (e.g., Levi et al. (2008b)), our transient unit-matching mechanism allows for temporary matching between certain qualified units of PB and OPT, tailored to our perishable inventory system. In addition, it parsimoniously matches these inventory units so that an optimal cost amortization can be achieved.

(2) Associated demand processes. Our analysis relies heavily on a notion of what-we-call *associated processes* formally defined in §2, which is crucial in analyzing a capacitated perishable inventory system with positive lead times. In contrast, the two closest works to ours, namely Chao et al. (2015) and Levi et al. (2008b), allow arbitrarily correlated demand processes. The worst-case analysis in these two references compares the aggregate ending inventory positions of any two feasible policies in each *single* period, which is determinable (or measurable) by the information available in that period. However, with the positive lead times and/or finite ordering capacities in this paper, comparing the ending inventory positions in a single period is insufficient to amortize the marginal costs of PB against OPT. This calls for a more sophisticated *global* cost amortization, which requires our analysis to “peek into the future”. It turns out that the class of associated demand processes, which embodies a notion of positive demand correlation, provides the delicate and precise future demand information necessary to yield the comparison results between PB and OPT. This makes our worst-case analysis drastically different from that in Chao et al. (2015), although a similar worst-case performance guarantee is established. For example, our approach relies on important submodularity/supermodularity properties of the backlogging cost as a function of ordering decisions and realizations of future demands, which we will establish for capacitated perishable inventory system with positive lead times.

Fortunately, associated demand processes are broad enough to include many demand processes of practical interest, including not only independent demand processes, but also many time series such as autoregressive (AR) and autoregressive moving average (ARMA) models (Box et al. (2015)), multiplicative auto-regression model (Levi et al. (2008a)), demand forecast updating processes such as martingale method of forecast evolution (MMFE) (Heath and Jackson (1994)) and demand processes with advance demand information (ADI) (Gallego and Özer (2001)), and economic-state driven demand process such as Markov modulated demand processes with stochastically monotone transition matrix (Sethi and Cheng (1997)), as special cases.

It is worth mentioning that almost all the approximation results for stochastic inventory systems hold for arbitrarily correlated demand processes. The only exception is Levi et al. (2008a) on lost-sales inventory systems with positive lead times, in which a worst-case performance guarantee was obtained only for three demand processes, namely independent demands, multiplicative autoregression demands, and AR(1) demands. The authors stated at the end of their paper: "*We believe that there are additional important demand structures for which the worst-performance guarantee can be shown. Providing a general characterization of the properties of the demand structure, required for the analysis to hold, is a very interesting future research direction.*" We find that the associated demand process is the right characterization of the demand process for the results in Levi et al. (2008a) for lost-sales non-perishable inventory systems with positive lead times. Hence as a by-product, this paper has also answered the open question raised in Levi et al. (2008a).

Our numerical study shows that the proposed policy performs much better than its theoretical worst-case performance guarantee. To simplify our presentation and improve the readability of this paper, whenever possible we shall refer to Chao et al. (2015) and Levi et al. (2008b). Thus, we will not repeat any argument used in their analysis. That is, all the technical proofs included in this paper are new and different from those in the literature.

1.1. Related Literature

The research on perishable inventory systems is pioneered by Veinott (1960), Van Zyl (1964) and Bulinskaya (1964). Most of the perishable inventory literature addresses various uncapacitated perishable inventory systems (see Chao et al. (2015) for a review of perishable systems without capacity constraints and zero lead times). The dominant paradigm in the existing literature has been to formulate and analyze these problems using dynamic programming. Many researchers characterized the structure of the optimal ordering policy with independent and identically distributed (i.i.d.) demands (e.g., Nahmias and Pierskalla (1973), Nahmias (1975), Fries (1975) and Cohen (1976)); and it has been shown that for the uncapacitated perishable systems, the optimal ordering quantity depends on both the age distribution of the current inventory and the remaining time period. The implication is that computing the optimal policy using brute-force dynamic programming is intractable due to state-dependent nature of the problem. Extensions of the basic uncapacitated model include Nahmias (1978), Nahmias and Pierskalla (1976), Deuermeyer (1979, 1980), and more recently, Chen et al. (2014) and Li et al. (2016).

Since computing the optimal inventory policy by dynamic programming is intractable even for the uncapacitated systems with zero lead times, many researchers turned to design effective heuristics (e.g., Nahmias (1976, 1977a,b), Nandakumar and Morton (1993), Cooper (2001)). The numerical results show that these heuristic policies perform well under i.i.d. demands for the uncapacitated systems. Recently, several researchers proposed heuristic policies for the joint inventory control and pricing models, and their heuristic policies perform well under i.i.d. demands (e.g., Chen et al. (2014) and Li et al. (2009)). Note that these studies focused on uncapacitated systems under i.i.d. demands, and none of the heuristics admits worst-case performance guarantees. Chao et al. (2015) also considered uncapacitated perishable inventory systems with zero lead time, and proposed two approximation algorithms that admit worst-case performance guarantees, which was then extended by Zhang et al. (2016) to incorporate setup costs. We are not aware of any effective heuristic policy proposed in the literature (with or without performance guarantees) for perishable inventory systems with positive lead times and finite ordering capacities.

There have been only a few papers on perishable inventory systems with positive lead times. Williams and Patuwo (1999, 2004) derived expressions for optimal ordering quantities of perishable inventory systems based on system recursions for a one-period problem with fixed lead time and lost-sales, and applied a nonlinear program to compute the optimal ordering quantities. Haijema et al. (2005, 2007) formulated the blood platelet production problem as a dynamic program, and combined it with a simulation method to compute for an optimal solution. Adachi et al. (1999) considered different selling prices of perishable inventory systems, and formulated the problem as a dynamic program to determine an optimal inventory policy that maximizes the expected profit. Chen et al. (2014) characterized the optimal policy for joint perishable inventory control and pricing problems with positive lead times. Sun et al. (2014) developed approximate dynamic program (ADP) heuristic algorithms for perishable inventory systems with positive lead times but without ordering capacity constraint. None of these studies provides any theoretical performance guarantee. Our work contributes to the literature by developing the first computationally efficient inventory control policy that admits a theoretical worst-case performance guarantee.

Our work is also closely related to the recent stream of literature on approximation algorithms for stochastic periodic-review inventory systems. Levi et al. (2007) initiated the concept of marginal cost accounting which associates the costs with decisions, and proposed a 2-approximation algorithm for the backlogging model. Subsequently, Levi et al. (2008b) introduced the concept of forced backlogging cost and proposed a 2-approximation algorithm for the capacitated model with backlogging. Levi et al. (2008a) proposed a 2-approximation algorithm for the lost-sales model with positive lead times. There have also been some recent studies on models with fixed costs (e.g., Levi and Shi (2013)) and multi-echelon systems (Levi et al. (2017)). All of these studies assume non-perishable products. As discussed above, Chao et al. (2015) analyzed the uncapacitated perishable inventory systems, and proposed two balancing policies with worst-case performance guarantees.

We end this review by mentioning some papers on capacitated inventory systems of non-perishable products. Federgruen and Zipkin (1986a,b) showed that a modified base-stock policy is optimal under both the average and discounted cost criteria. Tayur (1992), Kapuscinski and Tayur (1998), and Aviv and Federgruen (1997) derived the optimal policy under independent cyclical demands. Özer and Wei (2004) showed the optimality of modified base-stock policies in capacitated models with advance demand information. For capacitated perishable inventory systems, we are not aware of any literature that either characterizes the optimal policy or designs effective heuristic algorithms.

1.2. Structure and Notation

The rest of this paper is organized as follows. §2 presents the mathematical model and notations. §3 describes the marginal cost accounting scheme and the PB policy. §4 focuses on the worst-case analysis of the proposed policy. §5 summarizes our numerical results. Finally, §6 concludes the paper. Throughout the paper, we use increasing and decreasing in non-strict sense. We often distinguish between a random variable and its realization using a capital letter and a lowercase letter, respectively. We use " $:=$ " to mean "is defined as", and LHS and RHS as abbreviations for "left-hand side" and "right-hand side", respectively. The function $\mathbf{1}(A)$ is an indicator function taking value 1 if statement "A" is true and 0 otherwise; and for any real number x, we let $x^+ = \max\{x, 0\}$. For any sequence $x_1, x_2, \ldots$, we let $x_{[i,j]} = \sum_{k=i}^{j} x_k$ and $x_{[i,j)} = \sum_{k=i}^{j-1} x_k$, where the summation over an empty set is defined as 0. For any two real numbers a and b, we let $a \wedge b = \min\{a, b\}$.

2. Model Formulation

We provide the mathematical formulation of the capacitated periodic-review perishable stochastic inventory system with positive lead times. The planning horizon consists of T (possibly infinite) periods, indexed by $t = 1, 2, \ldots, T$. The ordering lead time is L periods and the product lifetime in stock is m periods (i.e., items perish after staying in stock for m periods), where $L \geq 1$ and $m \geq 1$ are deterministic integers. For each period t, the system has capacity u_t (≥ 0) in that the maximum order quantity in period t cannot exceed u_t.

Demand structure. The demands over the planning horizon, $D_1, \ldots, D_T$, are nonnegative random variables in a probability space $(\Omega, \mathscr{F}, P)$. At the beginning of each period t, there is an observed *information set* denoted by f_t, which contains the information accumulated up to period t, including not only the realized demands $d_1, \ldots, d_{t-1}$ in the first $t-1$ periods but also possibly some exogenous information. The information set $\{\mathscr{F}_t \mid 1 \leq t \leq T\}$ is the filtration generated by the process. We assume that all the relevant conditional expectations, given $\mathscr{F}_t$, are well defined.

The class of demand processes considered in this paper is called *associated processes*, which is defined below. Recall that random variables $D_1, \ldots, D_n$ are said to be associated (due to Essary et al. (1967)) if

$$\mathbf{Cov}(f(\mathbf{D}), g(\mathbf{D})) \geq 0, \quad \text{or equivalently} \quad \mathsf{E}[f(\mathbf{D})g(\mathbf{D})] \geq \mathsf{E}[f(\mathbf{D})]\mathsf{E}[g(\mathbf{D})],$$

for all non-decreasing (or non-increasing) functions $f(\cdot)$ and $g(\cdot)$ for which $\mathsf{E}[f(\mathbf{D})]$, $\mathsf{E}[g(\mathbf{D})]$, and $\mathsf{E}[f(\mathbf{D})g(\mathbf{D})]$ exist, where $\mathbf{D} = (D_1, \ldots, D_n)$.

Definition 1. A stochastic process $\{D_t; t = 1, 2, \ldots\}$ with filtration $\{\mathscr{F}_t, t \geq 1\}$ is associated if, conditioning on $\mathscr{F}_t$, the random variables $D_t, D_{t+1}, \ldots, D_s$ are associated for all $s > t \geq 1$.

This class of demand processes is broad enough to include many demand processes of practical interest, including as special cases not only independent demand processes, but also many time series such as autoregressive (AR) and autoregressive moving average (ARMA) models (Box et al. (2015)), multiplicative auto-regression model (Levi et al. (2008a)), demand forecast updating processes such as martingale method of forecast evolution (MMFE) (Heath and Jackson (1994)) and demand processes with advance demand information (ADI) (Gallego and Özer (2001)), and economic-state driven demand process such as Markov modulated demand processes with stochastically monotone transition matrix (Sethi and Cheng (1997) and Song and Zipkin (1993)). We provide the detailed descriptions of the aforementioned demand processes and the proof of Proposition 1 in the Appendix for interested readers.

Proposition 1. *AR, ARMA, multiplicative auto-regression model, MMFE, ADI, and Markov modulated demand processes modulated by stochastically monotone Markov chains, are associated processes.*

The associated demand process embodies a notion of positive correlation among demands, and it plays a crucial role in establishing the worst-case performance guarantee result in this paper. As mentioned in §1, it is also the right characterization of the demand process for the results in Levi et al. (2008a) for lost-sales non-perishable inventory systems with positive lead times. However, Chao et al. (2015) did not require such a demand structure for a simpler system with infinite capacity and zero lead time, and neither did Levi et al. (2008b) require it for capacitated non-perishable inventory systems; the results in both of these studies hold under arbitrarily correlated demand processes.

Cost structure. In each period t, $t = 1, \ldots, T$, four types of costs may occur: a unit ordering cost $\hat{c}$, a unit holding cost $\hat{h}$ for on-hand inventory and a unit holding cost $\tilde{h}$ for pipeline inventory (see Janakiraman and Roundy (2004) for a similar holding cost structure), a unit backlogging cost $\hat{b}$ for unsatisfied demand, and a unit outdating cost $\hat{\theta}$ for expired products. The outdating cost can be negative, and in that case it is interpreted as unit salvage value. There is also a one-period discount factor α, with $0 < \alpha \leq 1$ when $T < \infty$ and $0 < \alpha < 1$ when $T = \infty$. Following Nahmias (1975) and also for mathematical convenience, we assume that any remaining inventory at the end of the planning horizon can be salvaged with a return of $\alpha^{-L}(\hat{c} + \sum_{i=0}^{L-1} \alpha^i \tilde{h})$ per unit and unsatisfied demand can be satisfied by an emergency order with zero lead time at a cost of $\alpha^{-L}(\hat{c} + \sum_{i=0}^{L-1} \alpha^i \tilde{h})$ per unit. (Similar terminal conditions have been used in the study of non-perishable inventory systems, see, e.g., Veinott (1965).)

System dynamics. The sequence of events in each period t is as follows. First, at the beginning of period t, the order due in this period arrives and satisfies the backlogged demand, if any, as much as it can. Second, the firm observes the information set $f_t \in \mathscr{F}_t$ and the inventory vector

$$\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,m+L-1}),$$

where, for $i \in \{1, \ldots, m-1\}$, $x_{t,i}$ is the quantity of on-hand products whose remaining lifetime is i periods, $x_{t,m}$ is the quantity of products arriving at the beginning of period t less the quantity of the backlogged demand (if any), and $x_{t,m+j}$ is the quantity of pipeline inventory that will arrive in j periods, $j = 1, \ldots, L-1$. Note that $x_{t,m}$ can be positive or negative while all the other entries of $\mathbf{x}_t$ are non-negative, and if $x_{t,m}$ is negative, then $x_{t,1} = \ldots = x_{t,m-1} = 0$. Furthermore, for $j = 1, \ldots, L-1$, $x_{t,m+j}$ is the quantity ordered in period $t - L + j$. Denote x_t as the total on-hand inventory level in period t, i.e., $x_t = \sum_{i=1}^{m} x_{t,i}$. For simplicity, we assume that the inventory system is initially empty at the beginning of period 1, i.e., $x_{1,i} = 0$, for all $i = 1, \ldots, m+L-1$; but our analysis and results can be extended to the case with an arbitrary initial state.

Third, the firm places an order with quantity $q_t \in [0, u_t]$, incurring an ordering cost $\hat{c}q_t$. Since the ordering lead time is L periods, the order will arrive at the beginning of period $t + L$. Each unit of pipeline inventory incurs a holding cost $\tilde{h}$ per period. Let y_t denote the inventory position in period t after the replenishment decision, which is equal to the on-hand inventory level plus the pipeline inventory, i.e., $y_t = \sum_{i=1}^{m+L-1} x_{t,i} + q_t$.

Fourth, the demand in period t, D_t, is realized and satisfied as much as possible by the on-hand inventory using the FIFO issuing policy, i.e., the oldest inventory is consumed first when demand arrives. At the end of period t, if $x_t \geq D_t$, then the excess inventory incurs a holding cost $\hat{h}(x_t - D_t)$; while if $x_t < D_t$, then the system incurs a backlogging cost $\hat{b}(D_t - x_t)$. In addition, if the inventory units with one-period remaining lifetime $x_{t,1} > D_t$, then $e_t := x_{t,1} - D_t$ units outdate and the system incurs an outdating cost $\hat{\theta} e_t$.

Finally, the system proceeds to the subsequent period $t+1$. By the definition of the inventory vector $\mathbf{x}_t$ and the FIFO issuing policy, the state transition from $\mathbf{x}_t$ to $\mathbf{x}_{t+1}$ when $L \geq 2$ is

$$
\begin{aligned}
&x_{t+1,k} = (x_{t,k+1} - (D_t - \sum_{i=1}^{k} x_{t,i})^+)^+, \qquad \text{for } k = 1, \ldots, m-1; \\
&x_{t+1,m} = x_{t,m+1} - (D_t - \sum_{i=1}^{m} x_{t,i})^+; \\
&x_{t+1,k} = x_{t,k+1}\,, \text{ for } k = m+1, \ldots, m+L-2; \\
&x_{t+1,k} = q_t\,, \text{ for } k = m+L-1.
\end{aligned}
$$

When $L = 1$, the state transition is slightly different from above in that the term $x_{t,m+1}$ needs to be replaced by q_t. We remark that the system dynamic equations above are for the model of positive lead times, and they do not reduce to those for the model in Chao et al. (2015) with zero lead time.

Objective and policy assessment. The expected total discounted cost incurred under a given policy P that orders q_t in period t can be written as

$$
\begin{aligned}
\mathscr{C}(P) = \mathsf{E}\Bigg[\sum_{t=1}^{T} \alpha^{t-1} \Big(\hat{c} q_t + \tilde{h}(Y_t - X_t) + \hat{h}(X_t - D_t)^+ + \hat{b}(D_t - X_t)^+ + \hat{\theta} e_t\Big) \\
- \alpha^{T-L} \Big(\hat{c} + \sum_{i=0}^{L-1} \alpha^i \tilde{h}\Big) \sum_{i=1}^{m-1} x_{T+1,i}\Bigg],
\end{aligned}
\qquad (1)
$$

where we assume $q_t := 0$ for $t = T-L+1, \ldots, T$ since these orders will not arrive before the end of the planning horizon.

Following a similar cost transformation to that in Chao et al. (2015), we can transform the model above to an equivalent one with zero unit ordering cost and zero pipeline inventory cost. That is, there is an equivalent system with only three cost parameters $h = \hat{h} + \alpha^{-L}(1-\alpha)\hat{c} + (\alpha^{-L} - 1)\tilde{h}$,

$b = \hat{b} - \alpha^{-L}(1-\alpha)\hat{c} - (\alpha^{-L}-1)\tilde{h}$, and $\theta = \hat{\theta} + \alpha^{1-L}\hat{c} + \sum_{i=0}^{L-1}\alpha^{-i}\tilde{h}$. We assume $b > 0$ and $\theta \geq 0$. Then, the expected total discounted cost in (1) can be rewritten as

$$\mathscr{C}(P) = \mathsf{E}\left[\sum_{t=1}^{T}\alpha^{t-1}\left(h(X_t - D_t)^+ + b(D_t - X_t)^+ + \theta e_t\right)\right] + \sum_{t=1}^{T}\alpha^{t-1-L}\Big(\hat{c} + \sum_{i=0}^{L-1}\alpha^i\tilde{h}\Big)\mathsf{E}\left[D_t\right]. \quad (2)$$

This will enable us to assume, without loss of generality, zero unit ordering cost and zero pipeline holding cost in the subsequent analysis. Note that, the quantities q_t, e_t and X_t all depend on the policy P; and whenever necessary, we shall make the dependency explicit, i.e., write them as q_t^P, e_t^P and X_t^P, etc.

The objective is to coordinate the sequence of orders that minimizes the expected total discounted system-wide cost over periods $1, 2, \ldots, T$. It is well-known that finding the exact optimal policy is computationally intractable. Thus, our focus is to design an easy-to-compute and near-optimal approximation algorithm. To measure the effectiveness of an approximation algorithm, say P, we define its performance measure by the ratio $\mathscr{C}(P)/\mathscr{C}(OPT)$, where $\mathscr{C}(OPT)$ is the expected cost under an optimal policy.

3. Proportional-Balancing Policy

We present the nested marginal cost accounting scheme in §3.1; based on this scheme, we develop the PB policy and present its theoretical worst-case performance guarantee in §3.2. A detailed discussion on the worst-case bound is provided in §3.3.

3.1. Nested Marginal Cost Accounting Scheme

We present the nested marginal holding, outdating and backlogging costs for the capacitated perishable inventory system with positive lead times, which extend the cost accounting schemes in Chao et al. (2015) and Levi et al. (2008b). The key idea underlying this scheme is to decompose the total cost into the marginal costs associated with individual ordering decisions (note that the traditional cost accounting scheme described in (2) above decomposes the total cost into costs by periods). That is, we associate the ordering decision in each period t with its marginal costs. These marginal costs include costs (associated with the decision) which may incur in multiple periods after the order arrives at the system; and they are only affected by future demands but not by future decisions. The marginal cost accounting scheme for the perishable inventory system exhibits a nested structure, due to the expanded state vector representing the age distribution of the on-hand inventory level and pipeline inventories.

Nested marginal holding and outdating cost accounting. Since the marginal holding and outdating costs are defined similarly to those in Chao et al. (2015), we only provide their definitions here and refer readers to that paper for details. For $t=1,\dots,T-L$, suppose the inventory vector of the system at the beginning of period t is $\mathbf{x}_t=(x_{t,1},\dots,x_{t,m+L-1})$ and a policy P orders q_t. The holding cost for these q_t units may be incurred in any period from $t+L$ (at which the order arrives at the system) to $t+m+L-1$ (after which the remaining inventory, if any, will perish) or T, whichever is smaller. Let $H_t^P(q_t)$ be the total discounted (to period 1) *marginal holding cost* incurred by these q_t units. To compute this cost, for $s=t,\dots,t+m+L-2$, we denote $B_{[t,s]}(\mathbf{x}_t)$ as the number of outdated units in periods $[t,s]$ given that the inventory vector at the beginning of period t is $\mathbf{x}_t$. With the convention that $B_{[t,t)}(\mathbf{x}_t)\equiv 0$, we can write $B_{[t,s]}(\mathbf{x}_t)$ recursively: for $s=t,\dots,t+m+L-2$,

$$B_{[t,s]}(\mathbf{x}_t)=\max\left\{\sum_{i=1}^{s-t+1}x_{t,i}-D_{[t,s]},B_{[t,s)}(\mathbf{x}_t)\right\}. \tag{3}$$

The marginal holding cost $H_t^P(q_t)$ can be written as

$$H_t^P(q_t)\ :=\ h\sum_{s=t+L}^{(t+m+L-1)\wedge T}\alpha^{s-1}\Big(q_t-(D_{[t,s]}+B_{[t,s-1]}(\mathbf{x}_t)-\sum_{j=1}^{m+L-1}x_{t,j})^+\Big)^+. \tag{4}$$

Note that $H_t^P(\cdot)$ depends on $\mathbf{x}_t$ but for simplicity we make the dependency implicit. It is clear from (4) that $H_t^P(q_t)$ is increasing and convex in q_t, and that it is only affected by future demands but not by future decisions.

In contrast, the outdating cost for these q_t units may only incur in period $t+m+L-1$ if $t\le T-m-L+1$ while it is always zero if $t>T-m-L+1$ (since the ordered units will not expire within the planning horizon). Denote $\Theta_t^P(q_t)$ as the discounted (to period 1) *marginal outdating cost* incurred by these q_t units. Then, for $t\le T-m-L+1$,

$$\begin{aligned}\Theta_t^P(q_t)&:=\alpha^{t+m+L-2}\,\theta\,e_{t+m+L-1}\\&=\alpha^{t+m+L-2}\,\theta\,(q_t-(D_{[t,t+m+L-1]}+B_{[t,t+m+L-2]}(\mathbf{x}_t)-\sum_{j=1}^{m+L-1}x_{t,j})^+)^+;\end{aligned} \tag{5}$$

and for $t>T-m-L+1$, $\Theta_t^P(\cdot)\equiv 0$. It is clear from (5) that $\Theta_t^P(q_t)$ is increasing and convex in q_t.

Nested marginal backlogging cost accounting. In contrast to the uncapacitated systems, the key challenge in defining marginal backlogging costs for capacitated systems is that under-ordering in one period may result in backlogging costs not only in the period when the order arrives but also in later periods due to capacity constraints (since the system may be unable to clear the backlogged demand even if ordering full capacity in the next period). For capacitated systems with non-perishable products, Levi et al. (2008b) overcame this challenge by introducing an important concept called the *forced backlogging cost*, which is defined as the difference between the backlogging cost of an under-ordering decision in a period and that of ordering full capacity. In what follows, we extend this concept to the capacitated systems with perishable products.

In the capacitated perishable inventory system, the order quantity q_t in each period t is constrained by a capacity u_t, i.e., $q_t \le u_t$, $t = 1, \ldots, T - L$. To define the forced backlogging cost, we first define the regular backlogging cost Π_s^P as the discounted (to period 1) backlogging cost in period s for policy P, i.e.,

$$\Pi_s^P := \alpha^{s-1} b(D_s - X_s^P)^+.$$

In addition, for policy P, we denote $P(t)$ as the policy that follows policy P until period t, but then orders full capacity for all the subsequent periods $t+1, \ldots, T$. Then, for any $s \ge t$, $\Pi_s^{P(t)} - \Pi_s^{P(t-1)}$ is the shortage cost in period s resulting from the decision made in period t, q_t, that could have been saved by ordering the capacity u_t in period t (for simplicity we often make the dependency on q_t implicit). This is referred to as the *forced backlogging cost* in period s due to the decision q_t made in period t.

For $t = 1, \ldots, T - L$, we define the *marginal backlogging cost* associated with the decision q_t in period t, denoted by $\widetilde{\Pi}_t^P(q_t)$, as the sum of the forced backlogging costs in all future periods. That is,

$$\widetilde{\Pi}_t^P(q_t) := \sum_{s=t}^{T} (\Pi_s^{P(t)} - \Pi_s^{P(t-1)}) = \sum_{s=t+L}^{T} (\Pi_s^{P(t)} - \Pi_s^{P(t-1)}), \tag{6}$$

where the second equality follows from $\Pi_s^{P(t)} \equiv \Pi_s^{P(t-1)}$ for any $t \le s < t + L$, as the order placed in period t will not arrive until period $t + L$.

The following proposition summarizes some important properties on the marginal backlogging cost, which will be used in our subsequent analysis. Its proof is given in Appendix B.

Proposition 2. *For $t = 1, \ldots, T$ and for each sample path, the marginal backlogging cost $\widetilde{\Pi}_t^P(q_t)$ is decreasing and convex in the order quantity q_t, decreasing in vector $\mathbf{x}_t$, and increasing in demands $(D_t, \ldots, D_T)$.*

Since the marginal backlogging cost is defined as the sum of the differences between two regular backlogging costs with different ordering quantities, Proposition 2 leads to important submodularity/supermodularity results for the regular backlogging cost function: the backlogging cost function in any future period s $(> t)$ is supermodular in the order quantity q_t and the inventory vector $\mathbf{x}_t$

and submodular in the order quantity q_t and the future demands $(D_t,\dots,D_T)$. The latter property will play a central role in comparing the expected costs of our proposed policy and an optimal policy under associated demand processes.

To compute the marginal backlogging cost $\widetilde{\Pi}_t^P(q_t)$, we need to extend the notation $B_{[t,s]}(\mathbf{x}_t)$ to the case when $s \ge t+m+L-1$. Specifically, for any $t \le s$, we denote $B_{[t,s]}(\mathbf{x}_t, q_t)$ as the number of outdated units in periods $[t,s]$ under the policy that orders q_t in period t and full capacity in all subsequent periods, given that the inventory vector at the beginning of period t is $\mathbf{x}_t$. Note that $B_{[t,s]}(\mathbf{x}_t, q_t) \equiv B_{[t,s]}(\mathbf{x}_t)$ when $s \le t+m+L-2$, since the inventory is issued in an FIFO manner and the products ordered in and after period t will not outdate before period $t+m+L-1$. When $s > t+m+L-2$, we can compute $B_{[t,s]}(\mathbf{x}_t, q_t)$ recursively as follows:

$$B_{[t,s]}(\mathbf{x}_t, q_t) = \max\left\{ \sum_{i=1}^{m+L-1} x_{t,i} + q_t + \sum_{i=t+1}^{s-m-L+1} u_i - D_{[t,s]}, B_{[t,s)}(\mathbf{x}_t, q_t) \right\}. \tag{7}$$

For $s \ge t+L$, one can verify that

$$\Pi_s^{P(t)} = \alpha^{s-1} b \left(D_{[t,s]} + B_{[t,s)}(\mathbf{x}_t, q_t) - \Big(\sum_{i=1}^{m+L-1} x_{t,i} + q_t + \sum_{i=t+1}^{s-L} u_i \Big) \right)^+ .$$

Then, the forced backlogging cost in period s due to decision q_t in period t is

$$\begin{aligned} \Pi_s^{P(t)} - \Pi_s^{P(t-1)} = \alpha^{s-1} b \Bigg[& \left(D_{[t,s]} + B_{[t,s)}(\mathbf{x}_t, q_t) - \Big(\sum_{i=1}^{m+L-1} x_{t,i} + q_t + \sum_{i=t+1}^{s-L} u_i \Big) \right)^+ \\ & - \left(D_{[t,s]} + B_{[t,s)}(\mathbf{x}_t, u_t) - \Big(\sum_{i=1}^{m+L-1} x_{t,i} + \sum_{i=t}^{s-L} u_i \Big) \right)^+ \Bigg]. \end{aligned} \tag{8}$$

Consequently, the marginal backlogging cost $\widetilde{\Pi}_t^P(q_t)$ can be computed in a nested fashion using equations (6) to (8).

System cost decomposition. We show that the marginal costs constitute a decomposition of the total system cost. To that end, summing up $\widetilde{\Pi}_t^P$ of (6) over t, we obtain

$$\begin{aligned} \sum_{t=1}^{T-L} \widetilde{\Pi}_t^P &= \sum_{t=1}^{T-L} \sum_{s=t+L}^{T} (\Pi_s^{P(t)} - \Pi_s^{P(t-1)}) = \sum_{s=1+L}^{T} \sum_{t=1}^{s-L} (\Pi_s^{P(t)} - \Pi_s^{P(t-1)}) \\ &= \sum_{s=1+L}^{T} (\Pi_s^{P(s-L)} - \Pi_s^{P(0)}) = \sum_{s=1}^{T} \alpha^{s-1} b (D_s - X_s^P)^+ - \sum_{s=1}^{T} \Pi_s^{P(0)}, \end{aligned}$$

where the last equality holds since for any $s \ge L+1$, $\Pi_s^{P(s-L)}$ is the backlogging cost in period s under policy P, and the backlogging costs from periods 1 to L are the same for all policies (as the first order arrives in period $L+1$). Thus, the expected total cost under a feasible policy P in (2) can be rewritten as

$$\mathscr{C}(P) = \mathsf{E}\left[\sum_{t=1}^{T-L}(H_t^P + \Theta_t^P + \widetilde{\Pi}_t^P)\right] + \mathsf{E}\left[\sum_{t=1}^{T}\left(\Pi_t^{P(0)} + \alpha^{t-1-L}\Big(\hat{c} + \sum_{i=0}^{L-1}\alpha^i \tilde{h}\Big)D_t\right)\right].$$

Since the last term is independent of the policy P and nonnegative, we can ignore it in comparison and optimization among different policies. Thus, if ignoring the constant term, we can decompose the total system cost into the sum of the marginal holding, backlogging, and outdating costs.

3.2. Proportional-Balancing Policy

With the nested marginal cost accounting scheme developed in §3.1, we are ready to present our approximation algorithm. For each period $t = 1, \ldots, T-L$, given the inventory vector $\mathbf{x}_t$ and the observed information set f_t, the proportional-balancing (PB) policy orders $q_t^{PB} = q_t$ that balances a proportion of the expected marginal holding and outdating costs with the expected marginal backlogging cost, as follows:

$$\beta \mathsf{E}[H_t^{PB}(q_t) + \Theta_t^{PB}(q_t) \mid f_t] = \mathsf{E}[\widetilde{\Pi}_t^{PB}(q_t) \mid f_t], \tag{9}$$

where the balancing coefficient $\beta \in (0,1]$ is defined as

$$\beta := \begin{cases} 1, & \text{if} \quad m = 1; \\ \frac{\sum_{i=0}^{m-1}\alpha^{-i}h + \theta}{\sum_{i=-m+1}^{m+L-2}\alpha^i h + \theta}, & \text{if} \quad m > 1. \end{cases} \tag{10}$$

As noted earlier, the LHS of (9) is an increasing convex function of the order quantity q_t, which equals zero when $q_t = 0$ and approaches infinity when q_t tends to infinity. On the other hand, Proposition 2 shows that the expected marginal backlogging cost, i.e., the RHS of (9), is a decreasing convex function of the order quantity q_t, which takes a nonnegative value when $q_t = 0$ and equals zero when $q_t = u_t$. Since we assume that q_t can take any nonnegative real value and both functions are continuous, the quantity q_t that satisfies (9) is well-defined, and can be efficiently computed using bisection search methods.

For the special case that the system has an infinite capacity in each period (i.e., $u_t = +\infty$ for all t), one can easily verify from (6) and (8) that the marginal backlogging cost $\widetilde{\Pi}_t^P(q_t)$ reduces to the regular backlogging cost Π_{t+L}^P in period $t+L$. Consequently, in the uncapacitated system, the PB policy orders $q_t^{PB} = q_t$ that satisfies the following equation:

$$\beta \mathsf{E}[H_t^{PB}(q_t) + \Theta_t^{PB}(q_t) \mid f_t] = \mathsf{E}[\Pi_{t+L}^{PB}(q_t) \mid f_t]. \tag{11}$$

It is worth mentioning that although our PB policy is built on the ideas in Chao et al. (2015) and Levi et al. (2008b), the balancing coefficient β in (10) is much different from those in the above two studies. In fact, even for systems with zero lead times (i.e., $L=0$), our balancing coefficient β is different from that in Chao et al. (2015), and it leads to a tighter provable worst-case performance guarantee than that in Chao et al. (2015). More importantly, as will be seen in §4, our worst-case analysis requires different approaches and several new techniques, which departs significantly from the previous studies.

The following theorem presents the worst-case performance guarantee of the PB policy, which is the main theoretical result and contribution of this paper.

THEOREM 1. *For any associated demand process, the PB policy for capacitated perishable inventory control problems with m periods of lifetime and L periods of lead time has a worst-case performance guarantee of $\gamma := 1 + 1/\beta$ where β is defined in (10), i.e., for each instance of the problem, the expected cost of the PB policy is at most γ times the expected cost of an optimal policy.*

In the next subsection, we elaborate on the performance guarantee parameter γ as the system parameters vary.

3.3. Discussion of γ

Theorem 1 shows that, when $m=1$, the PB policy has a worst-case performance guarantee of 2 for an arbitrary positive lead time L. When $m \geq 2$, the theoretical worst-case bound γ is greater than or equal to 2. In the following proposition, we provide a necessary and sufficient condition for $\gamma \leq 3$ (i.e., the PB policy is a 3-approximation algorithm). The proofs of the statements in this subsection are given in Appendix C.

PROPOSITION 3. *Suppose $m \geq 2$. The worst-case bound $\gamma \leq 3$ if and only if*

$$\Gamma(\alpha,m,L)\hat{h} \leq \hat{\theta} + \alpha^{1-L}(\alpha^{-m} + \alpha^{m+L-2} - 1)\left(\hat{c} + \sum_{i=0}^{L-1}\alpha^i\tilde{h}\right), \tag{12}$$

where

$$\Gamma(\alpha,m,L) := \sum_{i=1}^{m+L-2}\alpha^i - \sum_{i=0}^{m-1}\alpha^{-i}. \tag{13}$$

We show that the condition (12) is mild and can be satisfied in most practical scenarios. First, it is satisfied when $\Gamma(\alpha,m,L) \leq 0$ which happens, for example, if $L \in \{1,2\}$. Second, when $\Gamma(\alpha,m,L) > 0$, a sufficient condition for (12) is

$$\Gamma(\alpha,m,L)(\hat{h} - \alpha^{-m-2}\tilde{h}) \leq \hat{\theta} + \alpha^{-1}\hat{c}, \tag{14}$$

which clearly holds when $\hat{h} \leq \alpha^{-m-2}\tilde{h}$. So let us consider the case when $\hat{h} > \alpha^{-m-2}\tilde{h}$. One can verify from (13) that $\Gamma(\alpha,m,L)$ is increasing in L and decreasing concave in m. In addition, it is increasing in α and $\partial^2\Gamma(1,m,L)/\partial\alpha^2 > 0$ when $L > 2$, indicating that $\Gamma(\alpha,m,L)$ is convex and increasing rapidly in α when α is close to one. To illustrate these, Table 1 below provides the values of $\Gamma(\alpha,m,20)$ for different values of α and m.

Table 1 **The threshold $\Gamma(\alpha,m,20)$ for different values of α and m**

	$\alpha = 0.91$	$\alpha = 0.93$	$\alpha = 0.95$	$\alpha = 0.97$
$m = 2$	6.48	8.10	10.14	12.72
$m = 3$	5.41	7.16	9.37	12.18
$m = 4$	4.21	6.12	8.53	11.60
$m = 5$	2.86	4.97	7.61	10.97

Table 1 shows that even when the lead time is as long as 20 periods, $\Gamma(\alpha,m,20)$ is less than 13 with $\alpha \leq 0.97$. We note that in most practical scenarios the lead time L should be small or moderate in length, and typically less than 20 periods. Hence, (14) holds when $\hat{h} - \alpha^{-m-2}\tilde{h}$ is small while $\hat{\theta} + \alpha^{-1}\hat{c}$ is relatively large. Indeed, these conditions tend to hold in most practical scenarios. As noted in prior studies (see, e.g., Axsäter (2007) and Nahmias and Olsen (2015)), the *annual* holding cost primarily consists of the capital cost and is typically less than 40% of the ordering cost. Hence, the unit *per-period* on-hand holding cost $\hat{h}$ would be a very small percentage (less than 40%/365 = 0.011% if the review period is a day) of the unit ordering cost $\hat{c}$, not to mention the difference between $\hat{h}$ and $\alpha^{-m-2}\tilde{h}$. Furthermore, for perishable products, the outdating cost $\hat{\theta}$ is usually either positive (due to the cost to dispose of the outdated products) or a small negative value relative to $\hat{c}$ (as the salvage value of the outdating product is rather small compared with its ordering cost). With these, we expect condition (14) to be satisfied in most practical situations and hence $\gamma \leq 3$.

Thus far we have discussed that γ is less than 3 for most cost parameters of practical interest. However, we want to point out that γ could be potentially very large for certain rare parameter settings. Let us take a closer look at γ. Since both h and θ are non-negative, one can easily verify that γ is upper bounded by $2 + (m + L - 2)\alpha/m$. Moreover, when $\alpha < 1$, γ is also upper bounded by $2 + 1/(\alpha^{-m} - 1)$. Since α^{-m} increases exponentially in the lifetime m when $\alpha < 1$, the worst-case bound γ approaches 2 when m becomes large. In addition, a necessary (but not sufficient) condition for γ to be large is when α is very close to 1 *and* the ordering lead time L is much longer than the product lifetime m. From the discussions above, a large γ also requires that the on-hand inventory holding cost $\hat{h}$ be much larger than the pipeline inventory holding cost $\tilde{h}$ and represents a large percentage of the ordering cost $\hat{c}$, or the outdating cost $\hat{\theta}$ is comparable to the negative of the ordering cost $\hat{c}$. We expect that very rare practical situations could simultaneously satisfy all these conditions (for example, it seems hard to find a real perishable-product example with a very long ordering lead time L but a short lifetime m). For those rare situations, γ could be large when the lead time L grows unboundedly. We also remark that the PB policy becomes less efficient in computation when the lead time L is very long, as in this case it would be time-consuming to evaluate the *expected* marginal costs.

Remark. For non-perishable inventory systems, a long ordering lead time is possible. Several recent papers (e.g., Goldberg et al. (2016), Xin and Goldberg (2016, 2017)) have shown that simple heuristics (e.g., constant-order policies) can be asymptotically optimal for non-perishable inventory systems with lost-sales or dual-sourcing when the ordering lead time tends to infinity. However, because lead times in perishable inventory systems are typically not long, we will not consider that scenario in our system.

We emphasize that the result in Theorem 1 is a theoretical worst-case bound. We have conducted an extensive numerical study in §5 and it shows that the PB policy performs much better than the worst-case bound.

4. Worst-Case Performance Analysis

In this section, we outline the proof of our main result (i.e., Theorem 1). To better understand how our approach is related to but different from those in the existing literature, we start by reviewing the general procedure for performing worst-case analysis of approximation algorithms for stochastic inventory systems, pioneered by Levi et al. (2007) and developed further by several subsequent studies, to bound the ratio of the expected total cost of a balancing policy to that of an optimal policy. Since any order placed after period $T-L$ will not arrive before the end of the planning horizon, it suffices to focus on policies that do not order after period $T-L$.

There are two major steps in the procedure. The first step is to construct, for each sample path, a rule that partitions the set of periods $\{1,\ldots,T-L\}$ into a subset $\mathscr{T}_H$ and its complementary subset $\mathscr{T}_\Pi$, and show that the marginal holding (respectively, backlogging) costs associated with the decisions made in periods $\mathscr{T}_H$ (respectively, $\mathscr{T}_\Pi$) under the balancing policy is upper bounded by the total marginal holding (respectively, backlogging) cost under OPT. The second step is to establish the comparison result on the expected total marginal costs on $\{1,\ldots,T-L\}$ between the balancing policy and OPT by taking conditional expectations over the filtration $\mathscr{F}_t$ on the sample-path results established in the first step. Almost all partition rules for $\mathscr{T}_H$ in the existing literature, including Chao et al. (2015) and Levi et al. (2008b), have been based on comparing each period's inventory positions (after ordering) under the balancing policy and OPT. Under this partition, $\mathbf{1}(t\in\mathscr{T}_H)$ is measurable with respect to the filtration $\mathscr{F}_t$, and the second step can go through readily. The only exception is Levi et al. (2008a), who studied the lost-sales inventory systems with positive lead times. For this system, a partition of periods based on comparing inventory positions under two policies provides insufficient information for sample-path analysis desired in step one. To overcome this difficulty, the authors constructed a different partition under which $\mathbf{1}(t \in \mathscr{T}_H)$ is unmeasurable with respect to $\mathscr{F}_t$ to facilitate the sample-path analysis but they established their result only for independent, multiplicative autoregression, and AR(1) demand processes.

For perishable systems with positive lead times and/or finite ordering capacities, the partition based on comparing the inventory positions also provides insufficient information for sample-path analysis; hence, we have to construct a new partition with an unmeasurable $\mathbf{1}(t \in \mathscr{T}_H)$ with respect to $\mathscr{F}_t$. Since the partition is different, our sample-path analysis is much different from Chao et al. (2015) for perishable inventory systems with zero lead time and no capacity constraint and Levi et al. (2008b) for capacitated non-perishable inventory systems. Moreover, as $\mathbf{1}(t \in \mathscr{T}_H)$ under our partition is unmeasurable under $\mathscr{F}_t$, similar to Levi et al. (2008a) we also need to impose conditions on the demand process to go through step two. In contrast to Levi et al. (2008a) who only analyzed three specific demand processes as mentioned above, we identify the class of associated demand processes for carrying out the conditional expectation analysis, which is a broad class of processes including most of the demand processes of practical interest.

We first provide the analysis for the uncapacitated system in §4.1 since a simpler algebraic proof suffices in this case. Then, we analyze the capacitated system in §4.2. Since the algebraic approach used in §4.1 fails to work for this more general case, we develop a *transient unit-matching mechanism* to prove the result. This matching mechanism divides products into two categories, those that expire and those that meet demand. The partition of periods, the cost amortization, and the analysis of conditional expectation, are all different from those of Chao et al. (2015) for uncapacitated perishable systems with zero lead times and those of Levi et al. (2008b) for capacitated non-perishable systems.

4.1. System without Capacity Constraints

We first consider the case when the system has an infinite ordering capacity for each period.

Similar to the concept "*trimmed on-hand inventory level*" in Chao et al. (2015), we define *the trimmed inventory position*, denoted by $Y_{t,s}$, for any $1 \le t \le s \le t+m+L-1$, as the inventory position Y_s at the beginning of period s minus the total order quantity in periods $t+1, \ldots, s$. Clearly, when $s=t$, $Y_{t,t}$ is equal to Y_t, the conventional inventory position in period t after the ordering decision; and when $s=t+L$, $Y_{t,t+L}$ is equal to X_{t+L}, the total on-hand inventory level in period $t+L$. In addition, when $t \ge s-L$, all the orders from periods $t+1$ to s are pipeline inventory, so in this case $Y_{t,s}$ is the part of the *inventory position* at the beginning of period s that is ordered in period t or earlier. It can be seen that,

$$Y_{t,t} = Y_t = q_t + \sum_{\ell=1}^{m+L-1} x_{s,\ell},$$

and for $s - L \le t < s$,

$$Y_{t,s} = \sum_{\ell=1}^{m+L+t-s} x_{s,\ell},$$

while for $s - m - L + 1 \le t < s - L$, we have

$$Y_{t,s} = \sum_{\ell=1}^{m} x_{s,\ell} - \sum_{\ell=t+1}^{s-L} q_\ell.$$

We remark that the trimmed inventory position is different from the trimmed on-hand inventory level in Chao et al. (2015) in that it is allowed to be negative. This modification is suitable for the analysis of capacitated perishable systems with positive lead times, whereas the latter is not. From its definition, for any $t \le s' \le s \le t + m + L - 1$, we have

$$Y_{t,s} = Y_{t,s'} - D_{[s',s)} - e_{[s',s)}. \tag{15}$$

Given any realization of $f_T \in \mathscr{F}_T$, define the set $\mathscr{T}_H$ and its complementary set $\mathscr{T}_\Pi$ as follows:

$$\mathscr{T}_H = \left\{ t \le T - L : \exists s \in \{t, \ldots, t+L\}, Y_{t,s}^{OPT} > Y_{t,s}^{PB} \right\}, \tag{16}$$

$$\mathscr{T}_\Pi = \left\{ t \le T - L : \forall s \in \{t, \ldots, t+L\}, Y_{t,s}^{OPT} \le Y_{t,s}^{PB} \right\}. \tag{17}$$

It is important to note that the above partition of periods, i.e., (16) and (17), is very different from that in Chao et al. (2015) and Levi et al. (2008b), which only relies on comparing the total inventory levels or positions in a *single* period.

Comparison of sample-path costs. We compare the costs between PB and OPT along each sample path. For an arbitrary positive lead time $L > 0$, we have the following result. Its proof is based on a similar but more refined algebraic analysis of Chao et al. (2015), and is thus omitted.

PROPOSITION 4. *For each realization $f_T \in \mathscr{F}_T$, we have*

$$\sum_{t \in \mathscr{T}_H} H_t^{PB} \le \sum_{t=1}^{T-L} H_t^{OPT} + \frac{h}{\theta} \sum_{i=1}^{m+L-2} \alpha^i \sum_{t=1}^{T-L} \Theta_t^{OPT}; \tag{18}$$

$$\sum_{t \in \mathscr{T}_H} \Theta_t^{PB} \le \sum_{t=1}^{T-L} \Theta_t^{OPT}; \tag{19}$$

$$\sum_{t\in\mathscr{T}_\Pi} \Pi_{t+L}^{PB} \le \sum_{t=1}^{T-L} \Pi_{t+L}^{OPT}. \tag{20}$$

Note that for each perished unit ordered in period t, with $t<T-L-m+2$, it incurs an outdating cost $\alpha^{t+L+m-2}\theta$ and a total discounted holding cost $\sum_{i=t+L-1}^{t+L+m-2}\alpha^i h$. Thus, for any policy, we have

$$\sum_{i=0}^{m-1}\alpha^{-i}\frac{h}{\theta}\sum_{t=1}^{T-L}\Theta_t \le \sum_{t=1}^{T-L} H_t, \tag{21}$$

where the LHS of (21) is the total holding cost incurred by all the perished units and the RHS of (21) is the total holding cost incurred by all the units, along the same sample path.

Combining (18), (19), (21) and by some simple algebra, we obtain that, for $m\ge 2$,

$$\sum_{t\in\mathscr{T}_H}\left(H_t^{PB}+\Theta_t^{PB}\right) \le \frac{1}{\beta}\sum_{t=1}^{T-L}\left(H_t^{OPT}+\Theta_t^{OPT}\right). \tag{22}$$

For the special case with $m=1$, the inequality (22) with $\beta=1$ follows directly from the inequality (19) of Proposition 4 and the observation that only perished units incur one period holding cost. Hence, the inequality (22) holds for an arbitrary lifetime m.

Analysis of expected costs. Applying Proposition 4 and (22), we now show how the associated demand process allows us to bound the ratio of the expected costs of the two systems, and complete the proof of Theorem 1. For convenience, we denote

$$Z_t^{PB} := \beta\mathsf{E}[H_t^{PB}+\Theta_t^{PB} \mid \mathscr{F}_t] = \mathsf{E}[\Pi_{t+L}^{PB} \mid \mathscr{F}_t].$$

Recall that $\gamma = 1+1/\beta$. Then, we have

$$\mathscr{C}(PB) = \sum_{t=1}^{T-L}\mathsf{E}[H_t^{PB}+\Theta_t^{PB}+\Pi_{t+L}^{PB}] = \gamma\sum_{t=1}^{T-L}\mathsf{E}[Z_t^{PB}]. \tag{23}$$

Thus, it follows from (20) of Proposition 4 and (22) that

$$\begin{aligned}\mathscr{C}(OPT) &= \sum_{t=1}^{T-L}\mathsf{E}[H_t^{OPT}+\Theta_t^{OPT}+\Pi_{t+L}^{OPT}]\\ &\ge \sum_{t=1}^{T-L}\mathsf{E}\left[\mathsf{E}\left[\beta\mathbf{1}(t\in\mathscr{T}_H)(H_t^{PB}+\Theta_t^{PB})+\mathbf{1}(t\in\mathscr{T}_\Pi)\Pi_{t+L}^{PB} \mid \mathscr{F}_t\right]\right].\end{aligned} \tag{24}$$

From the definition of Z_t^{PB}, we have

$$Z_t^{PB} = \beta\mathsf{E}[\mathbf{1}(t\in\mathscr{T}_H)\mid\mathscr{F}_t]\cdot\mathsf{E}[H_t^{PB}+\Theta_t^{PB}\mid\mathscr{F}_t] + \mathsf{E}[\mathbf{1}(t\in\mathscr{T}_\Pi)\mid\mathscr{F}_t]\cdot\mathsf{E}[\Pi_{t+L}^{PB}\mid\mathscr{F}_t].$$

Then, to complete proving Theorem 1 or $\mathscr{C}(PB) \le \gamma \mathscr{C}(OPT)$, it suffices to show that for each period t and given $\mathscr{F}_t$, the following two inequalities hold:

$$\mathsf{E}[\mathbf{1}(t \in \mathscr{T}_H) \mid \mathscr{F}_t] \cdot \mathsf{E}[H_t^{PB} + \Theta_t^{PB} \mid \mathscr{F}_t] \le \mathsf{E}[\mathbf{1}(t \in \mathscr{T}_H)(H_t^{PB} + \Theta_t^{PB}) \mid \mathscr{F}_t], \tag{25}$$

$$\mathsf{E}[\mathbf{1}(t \in \mathscr{T}_\Pi) \mid \mathscr{F}_t] \cdot \mathsf{E}[\Pi_{t+L}^{PB} \mid \mathscr{F}_t] \le \mathsf{E}[\mathbf{1}(t \in \mathscr{T}_\Pi)\Pi_{t+L}^{PB} \mid \mathscr{F}_t]. \tag{26}$$

Since the lead time L is positive, the events $\{t \in \mathscr{T}_H\}$ and $\{t \in \mathscr{T}_\Pi\}$ depend not only on $\mathscr{F}_t$ but also on the future demands $D_{t+1}, \ldots, D_{t+L-1}$. Thus, $[\mathbf{1}(t \in \mathscr{T}_H) \mid \mathscr{F}_t]$ and $[\mathbf{1}(t \in \mathscr{T}_\Pi) \mid \mathscr{F}_t]$, as well as $H_t^{PB} + \Theta_t^{PB}$ and Π_{t+L}^{PB}, are random and dependent on the future demands. Conditioning on $\mathscr{F}_t$, it is seen from our analysis in §3.1 that $H_t^{PB} + \Theta_t^{PB}$ is decreasing in $(D_t, \ldots, D_{T-L})$ and Π_{t+L}^{PB} is increasing in $(D_t, \ldots, D_{T-L})$. In addition, Proposition 5 below shows that $\mathbf{1}(t \in \mathscr{T}_H \mid \mathscr{F}_t)$ is decreasing in $(D_t, \ldots, D_{t+L-1})$, and thereby $\mathbf{1}(t \in \mathscr{T}_\Pi \mid \mathscr{F}_t) = 1 - \mathbf{1}(t \in \mathscr{T}_H \mid \mathscr{F}_t)$ is increasing in $(D_t, \ldots, D_{t+L-1})$. Therefore, by the definition of associated random variables, the inequalities (25) and (26) are satisfied if the demand process $\{D_1, \ldots, D_T\}$ is associated.

Proposition 5. *For each period $t = 1, \ldots, T-L$ and any $\mathscr{F}_t$, $\mathbf{1}(t \in \mathscr{T}_H \mid \mathscr{F}_t)$ is decreasing in $(D_t, \ldots, D_{t+L-1})$; thus the inequalities (25) and (26) hold if the demand process $\{D_1, \ldots, D_T\}$ is associated.*

The proof of Proposition 5 is given in Appendix D. The key takeaway message is that the class of associated demand processes, which embodies a notion of positive demand correlation, provides the right future demand structure for analyzing our refined partition of $\mathscr{T}_H$ and $\mathscr{T}_\Pi$. As mentioned earlier, this class is broad enough to include not only independent demand processes, but also most of the correlated demand processes discussed in the inventory management literature.

4.2. System with Capacity Constraints

We outline the key ideas for the proof of Theorem 1 for the system with capacity constraints.

For each realization f_T, we still partition the periods $\{1, \ldots, T-L\}$ into two subsets. Since the marginal backlogging cost for systems with capacity constraints is based on the forced backlogging cost and it is more complicated than that for systems with infinite capacity, the sample-path comparison result on the marginal backlogging costs can no longer be established when $\mathscr{T}_\Pi$ is defined by (17). To compare the marginal backlogging costs, we shall impose an even stronger condition on $\mathscr{T}_\Pi$ and modify the partition of periods from (16) and (17) to

$$\mathscr{T}_H = \left\{t \le T-L : \exists s \in \{t, \ldots, (t+m+L-1) \wedge T\}, Y_{t,s}^{OPT} > Y_{t,s}^{PB}\right\}, \tag{27}$$

$$\mathscr{T}_\Pi = \left\{t \le T-L : \forall s \in \{t, \ldots, (t+m+L-1) \wedge T\}, Y_{t,s}^{OPT} \le Y_{t,s}^{PB}\right\}. \tag{28}$$

With such a partition, we shall establish the parallel results of (18)-(20) for the system with both positive lead times and capacity constraints. Indeed, (19) can be shown similarly as before for this more general model. However, the proofs of the other two, for comparing sample-path marginal holding costs and marginal backlogging costs respectively, cannot be amended to handle the more general model, and their proofs require new ideas and techniques. In the following two subsections, we shall prove these two results (under Propositions 6 and 7).

Comparison of marginal holding costs. We shall establish the following result on the marginal holding costs.

PROPOSITION 6. *For each realization $f_T \in \mathscr{F}_T$, the inequality* (18) *holds.*

The algebraic proof for comparing marginal holding costs fails to work for the capacitated system. Therefore, we develop a new technique called *transient unit-matching* mechanism. Our approach uses a refined classification of inventory units, based on whether they eventually meet demand or expire. We also need to further decompose the periods in $\mathscr{T}_H$, according to whether or not the starting inventory (that will eventually meet demand) in the PB system is higher than that in the OPT. Since we are only interested in the units that incur holding cost, we shall ignore the pipeline inventory units in PB that meet the current backlogs.

The analysis stems from the following simple observation: Given any feasible policy P and a sample-path realization f_T, each unit ordered either meets demand or expires (for convenience and for this proof only, we treat those units that remain at the end of the planning horizon as expired units). Thus, we can write

$$Y_t^P = \tilde{Y}_t^P + \hat{Y}_t^P,$$

where the *effective* inventory $\tilde{Y}_t^P$ is the part of inventory in Y_t^P that will meet future demand, and the *perishing* inventory $\hat{Y}_t^P$ is the part of inventory in Y_t^P that will eventually expire. Here if $Y_t^P < 0$, then we let $\hat{Y}_t^P = 0$. Similarly, we split the order q_t^P in period t into the effective inventory $\tilde{q}_t^P$ and the perishing inventory $\hat{q}_t^P$; and $q_t^P = \tilde{q}_t^P + \hat{q}_t^P$. This segmentation of inventory helps analyze the two systems; it will be seen that the dynamics of these two types of inventory are different and it is the perishing inventory that makes the analysis challenging.

For each sample-path realization f_T, we can match each effective unit in PB with an effective unit in OPT in a one-to-one correspondence, i.e., a unit k in the effective inventory of PB is paired to a unit k' in the effective inventory of OPT *if and only if* they meet the same demand unit. We call (k, k') a *permanent pair*. From the definition of $\mathscr{T}_H$ in (27), the imposed information is too weak to imply that for any period $t \in \mathscr{T}_H$, all the permanent pairs of $\tilde{q}_t^{PB}$ are ordered before or at

t by OPT. Thus, we proceed to further divide the set $\mathscr{T}_H$ into two subsets, based on the effective inventory positions, as follows:

$$\mathscr{T}_{H1} = \left\{ t \in \mathscr{T}_H : \ \tilde{Y}_t^{OPT} > \tilde{Y}_t^{PB} \right\}; \quad \mathscr{T}_{H2} = \left\{ t \in \mathscr{T}_H : \ \tilde{Y}_t^{OPT} \leq \tilde{Y}_t^{PB} \right\}.$$

Given any two feasible policies P1 and P2, we denote $\hat{\mathbf{H}}^{P1}(\tilde{q}_t^{P2})$ as the marginal holding cost under P1 by those units that form permanent pairs with units in $\tilde{q}_t^{P2}$ under P2. For the periods in $\mathscr{T}_{H1}$, we have the following result.

Lemma 1. *For each realization $f_T \in \mathscr{F}_T$, if $t \in \mathscr{T}_{H1}$, then $\hat{q}_t^{PB} = 0$; and*

$$\sum_{t \in \mathscr{T}_{H1}} H_t^{PB} \leq \sum_{t \in \mathscr{T}_{H1}} \hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB}). \tag{29}$$

Lemma 1 is intuitive: In each period $t \in \mathscr{T}_{H1}$, since OPT has more effective inventory (that eventually meet the demand) than PB, all units PB orders in period t are effective inventory and meet the demand. Further, OPT must order these units earlier than PB by a conventional one-to-one unit matching argument (e.g., Levi et al. (2007)), thereby incurring more holding costs. We relegate its proof to Appendix E.

Compared with $\mathscr{T}_{H1}$, the analysis of $\mathscr{T}_{H2}$ is much more complex. The difficulty lies in that, for a period $t \in \mathscr{T}_{H2}$, some pairing units of $\tilde{q}_t^{PB}$ in OPT may be ordered after period t. Thus, OPT may not necessarily incur more holding costs than PB in periods $\mathscr{T}_{H2}$. To compare the costs of the two systems, we shall use the marginal holding cost of the perishing units in OPT to bound the marginal holding costs of PB within $\mathscr{T}_{H2}$, and establish the following important result.

Lemma 2. *For each realization $f_T \in \mathscr{F}_T$,*

$$\sum_{t \in \mathscr{T}_{H2}} H_t^{PB} \leq \sum_{t \in \mathscr{T}_{H2}} \hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB}) + \sum_{t=1}^{T-L} H_t^{OPT}(\hat{q}_t^{OPT}) + \frac{h}{\theta} \sum_{i=1}^{m+L-2} \alpha^i \sum_{t=1}^{T-L} \Theta_t^{OPT}, \tag{30}$$

where with slight abuse of notation, we use $H_t^{OPT}(\hat{q}_t^{OPT})$ to represent the marginal holding cost of the perishing units $\hat{q}_t^{OPT}$ ordered by OPT in period t.

To prove Lemma 2, we develop a novel transient unit-matching mechanism that allows for temporary matching between certain qualified units of PB and OPT. The main idea is to *parsimoniously* pair part of the inventory of PB with the perishing inventory of OPT – each perishing inventory unit of OPT could cover the holding cost of multiple units in PB, thereby obtaining an optimal cost amortization.

First we take a closer look at $\mathscr{T}_{H2}$. Applying the identity (15) and letting $t' = \min\{s \geq t : Y_{t,s}^{OPT} > Y_{t,s}^{PB}\}$ and after some simple algebra, we can rewrite $\mathscr{T}_{H2}$ as

$$\mathscr{T}_{H2} = \left\{ t \in \mathscr{T}_H : Y_t^{OPT} - e_{[t,t')}^{OPT} > Y_t^{PB} - e_{[t,t')}^{PB} \text{ and } \tilde{Y}_t^{OPT} \leq \tilde{Y}_t^{PB} \right\}. \tag{31}$$

Note that $t' \in \{t, \ldots, (t+L+m-1) \wedge T\}$ always exists because $t \in \mathscr{T}_{H2} \subseteq \mathscr{T}_H$. Also note that using this new definition (31), we can ignore the holding cost impact caused by the perishing units $e_{[t,t')}^{PB}$ and $e_{[t,t')}^{OPT}$ within periods $t, \ldots, t'-1$ in our analysis, because for any feasible policy P, the q_t^P units ordered in period t will not expire before period t' since $t' \leq t + L + m - 1$.

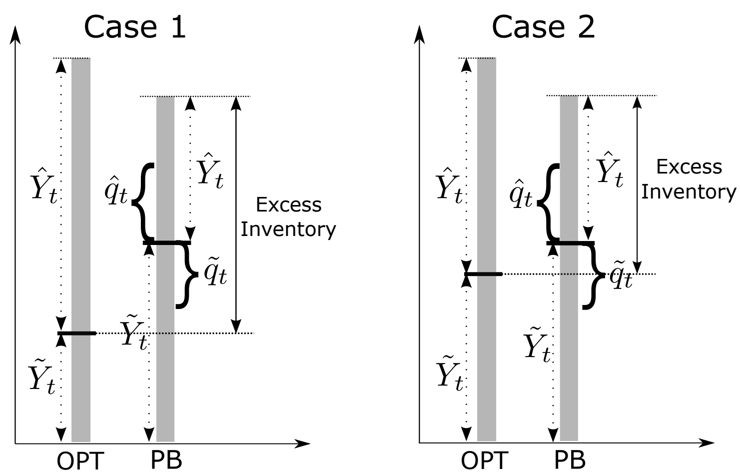

Figure 1 **Two cases for $t \in \mathscr{T}_{H2}$, after removing $e_{[t,t')}^{OPT}$ and $e_{[t,t')}^{PB}$, and pipeline inventories in PB and OPT that would meet current backlogs.**

We define the term *excess inventory* of PB to be the part of the inventory of PB in period t that is higher than the effective inventory $\tilde{Y}_t^{OPT}$ of OPT, then depending on the difference in excess inventory, there can only be two cases as illustrated in Figure 1. To analyze $\mathscr{T}_{H2}$, it suffices to consider the units in q_t^{PB} that belong to the excess inventory. This is because any inventory unit of q_t^{PB} not in the excess inventory of PB (i.e., the units ordered below the horizontal dash line) can always find its permanent pair in OPT.

For each realization f_T, if $t \in \mathscr{T}_{H2}$, then there are always sufficient perishing inventory units in OPT that can be paired with the excess inventory units in PB. We can pair the excess inventory in PB with the perishing inventory in OPT in a one-to-one correspondence. In particular, we pair a unit k in the excess inventory of PB with a unit l in the perishing inventory of OPT, and call (k, l) a *temporary pair*. The major difference from the conventional unit-matching mechanism (e.g., Levi et al. (2008b)) is that these temporary pairs may *break* over time, due to either expiration or the arrival of permanent pairs. Next, we formally describe the transient unit-matching mechanism.

Transient unit-matching mechanism. Denote the set of periods $\mathscr{T}_{H2}$ by $\{t_1, t_2, \ldots, t_q\}$ for some integer q. In period t_1, for each excess inventory unit in PB, we assign a perishing unit in OPT to be its temporary pair, according to the FIFO principle. Note that this assignment is valid since the perishing inventory of OPT is no smaller than the excess inventory of PB in period $t_1 \in \mathscr{T}_{H2}$.

Now, fix an arbitrary perishing unit l in OPT which is temporarily paired with some unit k_1 in the excess inventory of PB. We describe the pairing rule with respect to the perishing unit l within periods $\mathscr{T}_{H2}$ below. Suppose we are in period t_2, and recall the definition of t_2' from (31).

1) If l perishes before t_2', then l will not be used to pair with any other unit in the excess inventory of PB.

2) If l does not perish before period t_2' and k_1 is a perishing unit, then we have two sub-cases:

2a) If k_1 perishes before t_2', then the temporary pair (k_1, l) breaks and l becomes un-paired;

2b) if k_1 perishes in or after t_2', then l and k_1 are kept paired in period t_2.

3) If l does not perish before period t_2' and k_1 is an effective unit (thus k_1 has a permanent pair k_1' in OPT), then we have two sub-cases:

3a) If k_1' is ordered by OPT before or in period t_2, then the temporary pair (k_1, l) breaks and l becomes un-paired;

3b) if k_1' is ordered by OPT after period t_2, then l and k_1 are kept paired in t_2.

The procedure above is repeated for all such perishing unit l in OPT. For each un-paired unit in PB, we assign an un-paired unit in OPT to be its temporary pair, according to the FIFO principle. This assignment remains valid because the number of un-paired units in OPT is always larger than that in PB in period $t_2 \in \mathscr{T}_{H2}$. This concludes the pairing in period t_2. Then, we proceed to t_3, $t_4, \ldots$, until we complete the pairing procedure in period t_q. This completes the description of the matching mechanism.

The key in this matching mechanism is that a perishing unit l of OPT is used to pair with multiple units in the excess inventory of PB over its lifetime. That is, any perishing unit l corresponds to a set $\{(t_{l,1}, k_1), (t_{l,2}, k_2), \ldots, (t_{l,n'}, k_{n'})\}$, in which pair $(t_{l,i}, k_i)$, $t_{l,i} \in \mathscr{T}_{H2}$, $i = 1, \ldots, n'$, indicates that unit l of OPT is temporarily paired with unit k_i in the excess inventory of PB in period $t_{l,i}$.

Since we only need to amortize the marginal holding cost of excess inventory of PB ordered in periods $\mathscr{T}_{H2}$, we can remove all the pairs $(t_{l,i}, k_i)$ from the set $\{(t_{l,1}, k_1), (t_{l,2}, k_2), \ldots, (t_{l,n'}, k_{n'})\}$ if k_i is not ordered by PB in period $t_{l,i}$. For convenience we abuse the notation slightly to let $\{(t_{l,1}, k_1), \ldots, (t_{l,n}, k_n)\}$ denote the refined set of pairs after such pairs are removed, i.e., $t_{l,i} \in \mathscr{T}_{H2}$ and the unit k_i is ordered by PB in period $t_{l,i}$ for all $i = 1, \ldots, n$. We call this set the *mapping* of l. Figure 2 illustrates the *mapping* of a perishing unit l in OPT. In this example, the unit l has been in $\mathscr{T}_{H2}$ for 6 periods. In the first of these 6 periods, l is paired with a unit that is not ordered in t_2, and later l is paired with 3 different units in the excess inventory of PB. Thus, the mapping of l in

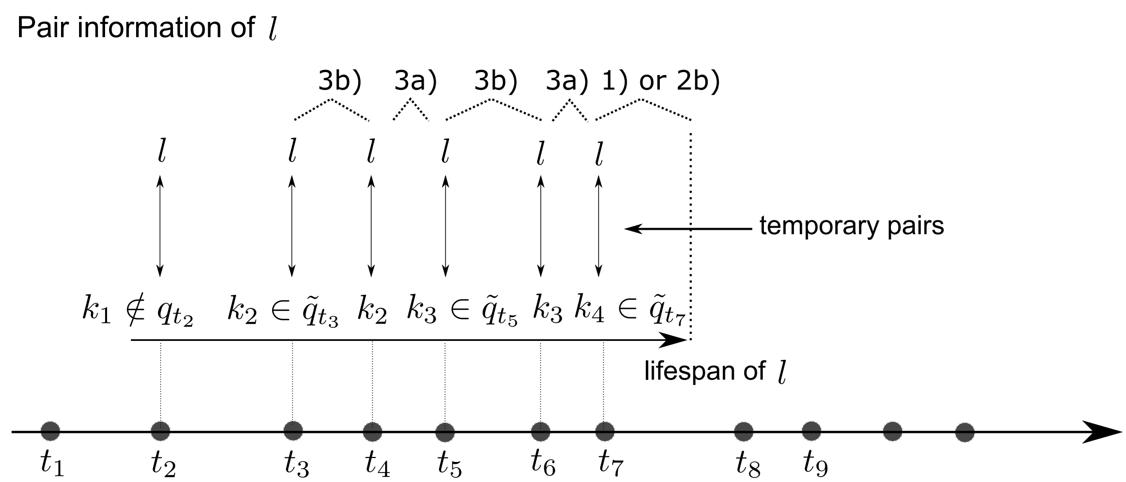

Figure 2 **Illustration of the** *mapping* **of a perishing unit** l **in OPT**

this example is $\{(t_3,k_2),(t_5,k_3),(t_7,k_4)\}$, i.e., $t_{l,1}=t_3$, $t_{l,2}=t_5$ and $t_{l,3}=t_7$. It can be seen that all the units $k_1,\ldots,k_n$ in the mapping of l are ordered no earlier than l. In addition, once l is paired with a "*fresh*" perishing unit in PB, it cannot be paired with any other unit in PB. Thus, when $n\geq 2$, $k_1,\ldots,k_{n-1}$ must be effective units of PB. The mapping could be empty if l has never been paired with any unit in PB during the planning horizon. Within $\mathscr{T}_{H2}$, any unit ordered by PB in the excess inventory *uniquely* belongs to the mapping of a perishing unit in OPT.

The following properties are satisfied by our transient unit-matching mechanism.

Lemma 3. *For each realization $f_T\in\mathscr{F}_T$, consider an arbitrary perishing unit l ordered by OPT in period t with a nonempty mapping $\{(t_{l,1},k_1),(t_{l,2},k_2),\ldots,(t_{l,n},k_n)\}$.*

(a) If $t<T-L-m+2$, then the total discounted holding cost of the units $k_1,k_2,\ldots,k_n$ in PB is at most that of the units $k_1',k_2',\ldots,k_{n-1}'$ in OPT plus $\sum_{i=-m+1}^{m+L-2}\alpha^i h/\theta$ times the discounted outdating cost of l;

(b) if $t\geq T-L-m+2$, then $n=1$, and k_1 is a perishing unit in PB incurring no more total discounted holding cost than unit l.

This key auxiliary result enables us to use the holding and outdating costs of unit l in OPT to parsimoniously cover the holding costs of multiple units of PB in its mapping. With Lemma 3 in place, we can prove Lemma 2. We delegate their detailed proofs to the Appendix E.

Once Lemmas 1 and 2 are established, adding (29) and (30) yields

$$\begin{aligned}\sum_{t\in\mathscr{T}_H} H_t^{PB} &\leq \sum_{t\in\mathscr{T}_H}\hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB})+\sum_{t=1}^{T-L}H_t^{OPT}(\hat{q}_t^{OPT})+\frac{h}{\theta}\sum_{i=1}^{m+L-2}\alpha^i\sum_{t=1}^{T-L}\Theta_t^{OPT}\\ &\leq \sum_{t=1}^{T-L}H_t^{OPT}+\frac{h}{\theta}\sum_{i=1}^{m+L-2}\alpha^i\sum_{t=1}^{T-L}\Theta_t^{OPT},\end{aligned}$$

where the last inequality is due to the fact that $\sum_{t=1}^{T-L} H_t^{OPT}(\hat{q}_t^{OPT})$ is the total holding cost of all the perishing units in OPT, and it has no overlap with $\sum_{t\in\mathscr{T}_H} \hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB})$, which is part of the total holding cost of the effective units of OPT. Hence their summation is bounded by the total marginal holding cost of OPT. This completes the proof of Proposition 6.

Comparison of marginal backlogging costs. We next compare the marginal backlogging costs of the two systems operating under policies PB and OPT respectively, and show that (20) is satisfied for perishable inventory systems with capacity constraints and positive lead times. We relegate the detailed proof of Proposition 7 to Appendix F.

PROPOSITION 7. *For each realization $f_T \in \mathscr{F}_T$, we have $\sum_{t\in\mathscr{T}_\Pi} \widetilde{\Pi}_t^{PB} \le \sum_{t=1}^{T-L} \widetilde{\Pi}_t^{OPT}$.*

After Propositions 6 and 7 are established for the system with capacity constraints, we can apply a similar analysis of the expected costs to that for the system without capacity constraints in §4.1 to establish Theorem 1 under associated demand processes. We omit the details for brevity.

5. Numerical Study

An important question is how the PB policy performs. In this section, we conduct a numerical study using demand distributions from Nahmias (1976, 1977b) and Nandakumar and Morton (1993), and our numerical results show that the approximation algorithm performs very well.

In the first subsection, we test the performance of the PB policy against the optimal solution. This requires us to focus on short ordering lead times. For relatively longer lead times, computing the optimal policies is not tractable; thus in §5.2 we compare the performance of our policy with the approximate dynamic program (ADP) heuristic policies developed in Sun et al. (2014).

5.1. Comparison with Optimal Policies

Following the literature, we compare the performance of an approximation policy P with that of the optimal policy OPT by their cost ratio $\mathscr{C}(P)/\mathscr{C}(OPT)$, and define the performance gap of an approximation policy P by

$$\%\ \text{Gap} = \left(\frac{\mathscr{C}(P)}{\mathscr{C}(OPT)} - 1\right) \times 100\%.$$

That is, the performance gap of an approximation policy is the percentage of the expected total cost increment of this policy over the planning horizon compared to the optimal minimum expected total cost.

We test three sets of examples and compare the results with the optimal solution: The first two have *i.i.d.* demands, with the first set having positive lead times and infinite capacities, while the second set having finite capacities and zero lead times. The third set of examples has correlated demands. Following the numerical studies in the literature on perishable inventory systems, the on-hand holding cost for all testing instances is normalized to $\hat{h}=1$, the pipeline holding cost $\tilde{h}=0$, and the discounted factor is set at $\alpha=0.95$. The planning horizon is $T=20$ periods. The unit ordering cost takes values from $\hat{c}\in\{0,5,10\}$, the backlogging cost is chosen from $\hat{b}\in\{5,10,15\}$, and the outdating cost $\hat{\theta}\in\{0,5,10,15\}$. For independent demands, we follow Nahmias (1976, 1977b) to test Erlang-2 distribution and exponential distribution with mean 10. For problems with positive lead times, we consider lifetime $m\in\{1,2\}$ and lead time $L\in\{1,2\}$; and for problems with capacity constraints, we consider $m\in\{2,3\}$ and $u\in\{10,15,20\}$. This gives a total of 720 testing problem instances for the first two sets of examples.

The numerical results for the problems with positive lead times are summarized in Table 2. The first column specifies lifetime m and ordering lead time L, and the second column is the demand distribution; the first row indicates the unit ordering cost $\hat{c}$ (Low, Medium and High represent the three ordering costs 0, 5 and 10 respectively). For given lead time, lifetime, demand distribution, and ordering cost, 9 combinations of backlogging costs and outdating costs are tested, and the two columns (mean and max) display the average and maximum performance gaps for the 9 problem instances. The overall performance is summarized in the last row and last two columns. We can see that the PB policy performs consistently well under various parameter settings, with an average gap of 2.49%.

The numerical results for the problems with capacity constraints are summarized in Table 3. In this table, the second column specifies the three possible capacities (Low, Medium and High stand for 10, 15 and 20 respectively), and other rows/columns have similar interpretations to those in Table 2. We can see that the largest gap occurs when the capacity is equal to the mean demand, and the average gap is 2.62%. Again it is seen that the results are consistent and the overall performance of the algorithm is quite good.

Table 2 Performance gap of PB for problems with positive lead times

	$\hat{c}$	Low		Medium		High		Overall	
(m,L)		mean	max	mean	max	mean	max	mean	max
$(1,1)$	Exp.	4.00%	5.51%	1.66%	2.42%	1.06%	1.80%	2.24%	5.51%
	Erlang-2	3.94%	6.30%	1.81%	2.68%	1.04%	1.60%	2.27%	6.30%
$(1,2)$	Exp.	3.75%	5.55%	2.14%	3.19%	1.27%	1.83%	2.39%	5.55%
	Erlang-2	2.71%	4.70%	2.28%	2.95%	1.35%	2.24%	2.12%	4.70%
$(2,1)$	Exp.	3.63%	5.28%	2.09%	2.88%	1.29%	2.10%	2.34%	5.28%
	Erlang-2	3.94%	4.88%	2.37%	3.34%	1.62%	3.51%	2.65%	4.88%
$(2,2)$	Exp.	4.78%	7.01%	2.48%	3.80%	1.65%	2.95%	2.97%	7.01%
	Erlang-2	4.45%	6.50%	2.86%	4.40%	1.63%	3.06%	2.98%	6.50%
Overall		3.90%	7.01%	2.21%	4.40%	1.36%	3.51%	2.49%	7.01%

Our third set of examples has Markov modulated demand processes with two states of the economy: 1 (bad) and 2 (good). When the state of the economy is 1, demand has mean 5, while in

Table 3 Performance gap of PB for problems with capacity constraints

		$\hat{c}$	Low		Medium		High		Overall	
m	u		mean	max	mean	max	mean	max	mean	max
2	Low	Exp.	8.16%	9.62%	3.21%	5.71%	1.84%	3.96%	4.40%	9.62%
		Erlang-2	7.30%	9.91%	2.89%	6.13%	1.20%	3.52%	3.80%	9.91%
	Medium	Exp.	2.91%	4.74%	1.45%	4.59%	0.63%	1.30%	1.66%	4.74%
		Erlang-2	3.19%	7.38%	1.23%	4.30%	0.60%	1.71%	1.68%	7.38%
	High	Exp.	1.93%	4.08%	1.21%	1.68%	1.27%	2.11%	1.47%	4.08%
		Erlang-2	2.09%	4.52%	1.42%	2.17%	1.46%	2.06%	1.66%	4.52%
3	Low	Exp.	6.56%	8.32%	3.01%	4.94%	1.28%	3.16%	3.62%	8.32%
		Erlang-2	6.96%	8.68%	3.27%	6.92%	1.65%	3.97%	3.96%	8.68%
	Medium	Exp.	4.23%	5.95%	2.01%	4.10%	1.48%	2.46%	2.57%	5.95%
		Erlang-2	4.72%	7.17%	2.32%	3.97%	1.55%	2.82%	2.86%	7.17%
	High	Exp.	2.74%	4.66%	1.69%	2.86%	1.48%	2.45%	1.97%	4.66%
		Erlang-2	2.42%	3.79%	1.87%	3.68%	1.09%	1.71%	1.79%	3.79%
Overall			4.43%	9.91%	2.13%	6.92%	1.29%	3.97%	2.62%	9.91%

state 2, demand has mean 10. The state of economy follows a Markov chain with a given transition matrix specified in Table 4. To compare with the optimal policy we consider $L = 1$ and $m = 2$. For brevity, we consider the problem without capacity constraint. Similar to the first two sets of examples, the demand is assumed to have exponential or Erlang-2 distribution whose mean is governed by the state of the economy. The cost parameters selected for testing are the same as those in the first set of examples. We have a total of 288 problem instances. For the first two transition matrices specified in Table 4, the Markov chains are stochastically monotone and thus the resulting demand processes are associated. The average performance gap of the PB policy is 2.82% and the maximum gap is 7.23%. For the last two transition matrices specified in Table 4, the Markov chains are not stochastically monotone and the resulting demand processes are not associated (they represent a scenario where, if the demand in a period is high/low, then the demand in the following period is more likely to be low/high, thus the demand tends to alternate between high and low). Note that the theoretical performance guarantee does not hold for these two examples. The average performance gap of the PB policy is 3.08% and the maximum gap is 11.43%. The performance of the PB policy, though still satisfactory, is indeed not as good as those under associated demand processes. In all the instances, we can see that the PB policy performs better with a larger ordering cost $\hat{c}$.

5.2. Comparison with Other Heuristics

For perishable inventory systems with longer lead times, the computation of their optimal policies is intractable. Hence we compare the performance of our policy against other heuristics. As far as we know, Sun et al. (2014) is the only paper that provides heuristics for perishable inventory systems with positive lead times (but without capacity constraints). A brief description of their

Table 4 Performance gap of PB for problems with correlated demands

		$\hat{c}$	Low		Medium		High		Overall	
Correlation	$\hat{b}$		mean	max	mean	max	mean	max	mean	max
$\begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$	Low	Exp.	3.96%	5.09%	2.61%	3.14%	2.14%	2.42%	2.90%	5.09%
	Medium		4.62%	5.02%	2.77%	3.26%	1.97%	2.20%	3.12%	5.02%
	High		3.45%	4.13%	2.36%	2.85%	1.81%	2.33%	2.54%	4.13%
	Low	Erlang-2	4.38%	5.45%	2.64%	3.37%	2.07%	2.34%	3.03%	5.45%
	Medium		4.32%	5.08%	2.67%	3.22%	1.78%	2.41%	2.92%	5.08%
	High		3.86%	5.52%	2.69%	3.79%	1.96%	2.99%	2.83%	5.52%
$\begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$	Low	Exp.	3.47%	4.49%	2.20%	2.77%	1.70%	2.26%	2.46%	4.49%
	Medium		4.07%	7.20%	2.36%	2.73%	1.27%	1.47%	2.56%	7.20%
	High		4.04%	7.23%	2.62%	4.13%	2.17%	2.74%	2.95%	7.23%
	Low	Erlang-2	3.39%	6.61%	2.27%	2.74%	1.56%	1.79%	2.41%	6.61%
	Medium		3.59%	6.59%	2.43%	3.23%	1.82%	2.71%	2.61%	6.59%
	High		4.71%	7.19%	3.26%	5.26%	2.41%	2.84%	3.46%	7.19%
Overall			3.99%	7.23%	2.57%	5.26%	1.89%	2.99%	2.82%	7.23%
$\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$	Low	Exp.	5.04%	7.51%	2.95%	3.87%	2.59%	3.41%	3.53%	7.51%
	Medium		5.42%	10.11%	2.03%	2.68%	1.60%	2.48%	3.02%	10.11%
	High		4.30%	8.95%	2.47%	3.70%	1.55%	2.22%	2.77%	8.95%
	Low	Erlang-2	4.73%	8.70%	2.24%	3.67%	2.01%	2.58%	2.99%	8.70%
	Medium		4.45%	9.47%	1.90%	2.68%	1.05%	1.48%	2.47%	9.47%
	High		5.33%	11.43%	2.55%	4.08%	1.40%	2.31%	3.09%	11.43%
$\begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$	Low	Exp.	5.63%	7.26%	2.67%	3.26%	1.79%	2.06%	3.36%	7.26%
	Medium		4.81%	7.28%	2.62%	3.69%	1.79%	2.04%	3.07%	7.28%
	High		5.04%	11.25%	2.19%	2.75%	1.50%	1.78%	2.91%	11.25%
	Low	Erlang-2	5.55%	7.16%	2.87%	3.64%	2.35%	2.65%	3.59%	7.16%
	Medium		5.26%	6.61%	2.18%	2.62%	1.54%	1.70%	2.99%	6.61%
	High		5.19%	9.75%	2.57%	3.25%	1.53%	1.96%	3.10%	9.75%
Overall			5.06%	11.43%	2.44%	4.08%	1.72%	3.41%	3.08%	11.43%

heuristics is provided below. Since their algorithm focuses on the infinite horizon problem, in this subsection we extend the planning horizon to $T=50$ and still keep $\alpha=0.95$. We analyze lead time $L=5$ and lifetimes $m=2$ and $m=4$. The unit holding cost $\hat{h}$ is again normalized to 1 and the unit backlogging and unit outdating costs $\hat{b}$ and $\hat{\theta}$ are both chosen from $\{5,10,15\}$. We set $\hat{c}=0$. Same as before we let the demand be i.i.d. exponential or Erlang-2 with mean 10. Thus in total we test $2\times3\times3\times2=36$ problem instances.

Sun et al. (2014) proposed three heuristic policies: MyopicMC, LP-Greedy policy, and T^L policy. MyopicMC is a myopic policy that in every period t minimizes the sum of expected holding and backlogging cost in period $t+L$ and the expected outdating cost in period $t+L+m-1$. The other two are approximate dynamic program (ADP) policies based on a linear program (LP) that solves the optimal value function for an infinite-horizon dynamic program. The key idea is to use a convex quadratic function to approximate the optimal value function. The main steps of computing these two policies are as follows. First, use the MyopicMC policy to construct a sample state space and then establish the objective function and constraints for the LP. Second, solve the LP to determine the coefficients of the approximate quadratic value function. Then, if directly solving the optimality equation with the approximate value function, then it gives the LP-Greedy policy; while if applying policy iteration on the resulting optimality equation for L times and then computing the optimal policy, it gives the T^L policy. Since the approximate value function is quadratic, each iteration has a closed form output, with the quadratic coefficients recursively updated for L times when computing the T^L policy.

In our numerical study, we follow Sun et al. (2014) to use the Myopic policy to generate 5000 sample states to establish the inputs for the LP. The LP is solved by CPLEX 12.4. After obtaining the approximate value functions, we use 50 periods of demands with 500 sample paths to evaluate the expected total discounted cost of the policies.

Following Levi and Shi (2013), our proposed PB policies can be parametrized to obtain a class of approximation policies, and instead of fixing the parameters, we can try different parameter values for a given problem instance, and then identify a parameter that empirically performs the best. In this subsection, we choose the balancing parameter γ' from $\gamma - 0.5$ to $\gamma + 0.5$, in a step size of 0.025 to find the best balancing parameter for every instance. For each problem instance, we randomly generate a set of demand data and divide it into in-sample data and out-of-sample data. We use the in-sample data to compute the best balancing parameter, and then use the out-of-sample data to evaluate its performance. We refer to this parametrized-PB policy as PPB.

The performances of PB and PPB and the three heuristics proposed in Sun et al. (2014) under the long lead time setting are summarized in Table 5. In this table, for each problem instance, we use **0%** to indicate the policy with the lowest expected cost, and the expected cost of an alternative policy is represented as the percentage above this lowest expected cost. As seen from the table, among the three heuristics proposed in Sun et al. (2014), T^L policy seems to perform the best, and its performance is comparable with that of our PPB policy.

6. Conclusion

In this paper, we propose the first approximation algorithm for perishable inventory systems with positive lead times and finite ordering capacities that admits a theoretical worst-case performance guarantee when the demand process is associated. This algorithm is computationally very efficient, and our numerical results show that it performs well. The class of associated demand processes is quite broad, including not only independent demand processes but also the commonly used time series models such as AR(p) and ARMA (p,q), multiplicative auto-regression model, demand forecast updating models such as MMFE and ADI, as well as demand models driven by economic state of the world, e.g., Markov modulated demand process driven by stochastically monotone transition matrices.

Table 5 Comparison of PB and PPB with other heuristics for problems with $L = 5$

		Exponential Distribution					Erlang-2 Distribution				
m	$(\hat{b},\hat{\theta})$	Myop	LPg	T^L	PB	PPB	Myop	LPg	T^L	PB	PPB
2	(5,5)	5.45%	3.55%	**0%**	2.87%	0.11%	3.33%	2.55%	0.17%	2.62%	**0%**
	(5,10)	6.39%	5.35%	0.22%	0.54%	**0%**	4.37%	4.21%	0.19%	0.85%	**0%**
	(5,15)	6.52%	5.33%	1.28%	0.14%	**0%**	4.48%	4.11%	0.65%	0.18%	**0%**
	(10,5)	3.98%	6.88%	**0%**	3.87%	0.11%	3.44%	17.27%	**0%**	4.56%	1.24%
	(10,10)	5.94%	7.96%	**0%**	1.67%	0.10%	4.54%	3.79%	**0%**	2.12%	0.54%
	(10,15)	6.89%	4.66%	0.61%	0.79%	**0%**	4.77%	3.09%	0.12%	0.97%	**0%**
	(15,5)	3.65%	10.37%	**0%**	3.93%	0.17%	1.68%	8.20%	0.02%	3.49%	**0%**
	(15,10)	5.44%	4.88%	**0%**	2.29%	0.27%	3.89%	6.13%	**0%**	2.63%	0.77%
	(15,15)	6.30%	4.28%	0.10%	1.15%	**0%**	4.23%	6.72%	0.15%	1.25%	**0%**
4	(5,5)	1.63%	1.90%	0.11%	3.64%	**0%**	0.02%	8.30%	0.44%	3.33%	**0%**
	(5,10)	2.93%	1.16%	**0%**	1.69%	0.19%	0.82%	4.50%	0.05%	1.71%	**0%**
	(5,15)	3.40%	0.86%	0.03%	0.62%	**0%**	1.42%	7.27%	**0%**	0.99%	0.04%
	(10,5)	0.96%	10.15%	1.80%	4.12%	**0%**	**0%**	15.15%	3.86%	4.39%	0.69%
	(10,10)	3.13%	3.30%	**0%**	2.83%	0.52%	0.83%	4.49%	**0%**	2.51%	0.31%
	(10,15)	4.00%	2.44%	**0%**	1.64%	0.26%	1.51%	7.24%	**0%**	1.76%	0.33%
	(15,5)	1.20%	8.67%	2.25%	4.26%	**0%**	**0%**	12.68%	4.04%	4.25%	0.77%
	(15,10)	2.98%	7.73%	**0%**	3.13%	0.69%	0.27%	3.07%	**0%**	2.17%	0.01%
	(15,15)	4.11%	5.40%	**0%**	2.18%	0.48%	1.14%	6.21%	**0%**	1.64%	0.29%

The theoretical performance bound in Theorem 1 is proved under the condition that the demand process is associated. When the demand process is not associated, then inequalities (25) and (26) may not hold and thus the theoretical bound may not be guaranteed. We tested the performances of the PB policy on some non-associated demand processes (more precisely, Markov modulated demand processes with non-monotone modulating Markov chains), and found that its performances on these examples are indeed not as good as those under associated demand processes, even though the results are reasonably good. We note that there is also a notion of negatively associated process (see Joag-Dev and Proschan (1983)), and it would be interesting and worthwhile to design approximation algorithms with worst-case performance guarantees for that setting.

An interesting future research direction is to investigate models with random ordering lead times and/or random product lifetimes. A common assumption in the literature is to assume that the lead time and the product lifetime are stochastic but non-crossing (e.g., Janakiraman and Roundy (2004)). This non-crossing assumption ensures that a later order would not arrive or expire earlier than an earlier order. Under this assumption, the marginal cost accounting scheme still holds, since the marginal costs of an ordering decision are not affected by future orders. We, therefore, conjecture that the performance guarantee γ could be slightly modified by replacing m and L by their respective lower/upper bounds of their supports.

Another interesting future research direction is to develop approximation algorithms for perishable inventory systems with lost-sales and positive lead times. For non-perishable inventory systems, Levi et al. (2008a) developed an approximation algorithm with a worst-case performance guarantee of 2. However, neither the period partition in this paper nor that of Levi et al. (2008a) works for the analysis of performance guarantee of perishable inventory systems with lost-sales and positive lead times. How to design a new partition scheme that allows for the worst-case performance analysis on this system remains an open question.

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APPENDIX

A: Associated Demand Processes

In this subsection, we provide some well-known demand processes that fall into the broader class of associated demand processes defined in §2 and also present the proof of Proposition 1.

(a) **Time-series models.** A demand process can often be modeled as a time series, so that at any time t, the demand in the next period is a linear combination of demands in the past and some other random variables. These time-series models include moving average (MA) model, autoregressive (AR) model, autoregressive moving average (ARMA) model, among others (see e.g., Box et al. (2015)). For example, in ARMA(p, q) model, the demand in period t is $D_t = d + \epsilon_t + \sum_{i=1}^{p} \psi_i D_{t-i} + \sum_{i=1}^{q} \theta_i \epsilon_{t-i}$, where ϵ_i, $i = 1, 2, \ldots$, are independent random variables, and ψ_i and θ_i are nonnegative constant coefficients.

(b) **Multiplicative autoregression model**. The demands are defined recursively via $D_t = N_t D_{t-1}$, where N_t, $t = 1, 2, \ldots$, are independent positive random variables with mean 1. This process has been analyzed in Levi et al. (2008a) for lost-sales inventory systems with positive lead times.

(c) **Martingale method of forecast evolution (MMFE).** The class of MMFE demand models (see, e.g., Heath and Jackson (1994)) is defined as follows. At the beginning of period 1 there is an initial forecast $D_{1,s}$ for demand in period $s \geq 1$. The forecasts are updated in each period such that the forecast in period $t+1$ is equal to the forecast in period t plus a term of adjustment, i.e., $D_{t+1,s} = D_{t,s} + \epsilon_{t,s}$, $s \geq t$, where we assume that the adjustment terms $\epsilon_{t,s}$, $s \geq t \geq 1$, are independent of one another. The actual demand in period s is given by $D_s = D_{s+1,s}$. It is clear that the demand in period t can be written as $D_t = D_{1,t} + \sum_{k=1}^{t} \epsilon_{k,t}$, $t = 1, 2, \ldots$.

(d) **Advance demand information (ADI).** A firm may often receive orders from some customers ahead of time. Gallego and Özer (2001) proposed a demand model with advance demand information. In this model, the firm receives demand $D_{t,s}$ in period t for future period s, hence the total demand for a period t is $D_t = \sum_{u \leq t} D_{u,t}$, where $D_{s,t}$ are independent over s and t.

(e) **Markov modulated demand processes (MMDP).** It is conceivable that the customer demand in a period depends on the state of the economy in that period, and a Markov modulated demand process is such a model that has been used in the literature (see, e.g., Sethi and Cheng (1997) and Song and Zipkin (1993)). We need the following definition due to Daley (1968). A Markov chain $\{I_t; t \geq 1\}$ is stochastically monotone if $\{I_{t+1} \mid I_t = i\}$ is stochastically increasing in i, i.e., for all $i < j$ and all k, it holds that $\Pr\{I_{t+1} \geq k \mid I_t = i\} \leq \Pr\{I_{t+1} \geq k \mid I_t = j\}$. A Markov modulated demand process is defined as follows. There is a Markov chain $\{I_t; t \geq 1\}$ with state space $\{1, 2, \ldots, M\}$, where M may be infinity. If I_t takes value i in period t, then the demand for period t is drawn from distribution $F_i(\cdot)$, where F_i is stochastically increasing over i, i.e., $F_i(z) \geq F_j(z)$ for any z if $i < j$. This means, a better economy (larger j) implies stochastically larger demand. For each $i = 1, \ldots, M$, let $\{V_i(s); s = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables with distribution function $F_i(\cdot)$. A demand process $\{D_t;\, t = 1, 2, \ldots\}$ modulated by Markov chain $\{I_t; t \geq 1\}$ is defined by $D_t = V_{I_t}(t)$, $t = 1, 2, \ldots$.

Proof of Proposition 1. An important property of associated random variables is that, increasing functions of independent random variables are associated. Most time-series forecasting models, including the ones discussed above, as well as MMFE and ADI, have the property that conditioning on the process up to period t, future demands can be written as linear combinations of past demands and some other independent random variables with nonnegative coefficients, hence they are associated processes.

We then consider Markov modulated demand processes. Conditioning on I_t, and $I_{t+1}, \ldots, I_{t+s}$, the demands $V_{I_{t+1}}, \ldots, V_{I_{t+s}}$ are independent random variables, hence it follows from independent random variables are associated that for any increasing functions f and g,

$$\begin{aligned} &\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}] \\ \geq\ &\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]\, \mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]. \end{aligned} \tag{32}$$

Since V_{I_t} is stochastically increasing in I_t, it follows from f and g are increasing function that both $\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]$ and $\mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]$ are increasing functions of $I_{t+1}, \ldots, I_{t+s}$. It is known from Proposition 2.6 of Cohen and Sachrowitz (1993) that stochastically monotone Markov chains are associated. Applying the association property of $I_{t+1}, \ldots, I_{t+s}$ and taking conditional expectation with respect to $I_{t+1}, \ldots, I_{t+s}$ for given I_t, we obtain

$$\begin{aligned} &\mathsf{E}[\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]\, \mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]] \\ \geq\ &\mathsf{E}[\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]]\, \mathsf{E}[\mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t, I_{t+1}, \ldots, I_{t+s}]] \\ =\ &\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t]\, \mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t]. \end{aligned} \tag{33}$$

Taking conditional expectation on both sides of (32) with respect to $I_{t+1}, \ldots, I_{t+s}$ for given I_t and combining the result with (33), we obtain

$$\mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t] \geq \mathsf{E}[f(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t]\, \mathsf{E}[g(V_{I_{t+1}}, \ldots, V_{I_{t+s}}) \mid I_t].$$

This shows that, conditional on I_t, the random variables $V_{I_{t+1}}, \ldots, V_{I_{t+s}}$ are associated, or the Markov demand process modulated by a stochastically monotone Markov chain is associated. **Q.E.D.**

B: Proof of Proposition 2

As a preparation for proving Proposition 2, we first provide a lemma which shows how the auxiliary function $B_{[t,s]}(\mathbf{x}_t, q_t)$ depends on q_t.

LEMMA 4. *For $t \le s \le t+m+L-2$, $B_{[t,s]}(\mathbf{x}_t, q_t)$ is independent of q_t. For $s \ge t+m+L-1$,*

$$B_{[t,s]}(\mathbf{x}_t, q_t) = B_{[t,s]}(\mathbf{x}_t, 0) + (q_t - K(\mathbf{x}_t))^+, \tag{34}$$

where $K(\mathbf{x}_t) := \inf\{q \ge 0 \mid B_{[t,s]}(\mathbf{x}_t, q) - B_{[t,t+m+L-2]}(\mathbf{x}_t, q) > 0\}$.

Proof. Recall that $B_{[t,s]}(\mathbf{x}_t, q_t)$ is defined as the number of outdated units in periods $[t,s]$ under the policy that orders q_t in period t and full capacity in all subsequent periods, given that the inventory vector at the beginning of period t is $\mathbf{x}_t$. When $s \le t+m+L-2$, since the inventory is consumed in an FIFO manner and the ordered products in and after period t will not outdate before period $t+m+L-1$, it is clear that $B_{[t,s]}(\mathbf{x}_t, q_t)$ is independent of q_t.

Now suppose $s \ge t+m+L-1$. We first show that $K(\mathbf{x}_t)$ is well defined. By the definition of $B_{[t,s]}(\mathbf{x}_t, q_t)$, it follows that $B_{[t,s]}(\mathbf{x}_t, q) - B_{[t,t+m+L-2]}(\mathbf{x}_t, q)$ is the total amount of outdated products in the interval $[t+m+L-1, s]$ when the inventory vector is $\mathbf{x}_t$ at the beginning of period t and the system orders $q, u_{t+1}, \dots, u_T$ in periods $t, t+1, \dots, T$, respectively. It can be seen that this quantity is continuous and increasing in q and it is strictly positive when q is sufficiently large. Thus, $K(\mathbf{x}_t)$ is well defined, with $K(\mathbf{x}_t) \ge 0$.

We next prove the identity (34). From the definition of $K(\mathbf{x}_t)$, we know that for any $q_t > K(\mathbf{x}_t)$, product outdating occurs in at least one period in $t+m+L-1, t+m+L, \dots, s$. In addition, since the amount of products outdated in each of the periods $t+m+L-1, t+m+L, \dots, s$ is non-decreasing in the order quantity q_t, there exists one period, denoted by s', such that some products outdate in this period for any order quantity q_t with $q_t > K(\mathbf{x}_t)$. Therefore, when $q_t > K(\mathbf{x}_t)$, we must have

$$\sum_{i=1}^{m+L-1} x_{t,i} + q_t + \sum_{k=t+1}^{s'-m-L+1} u_k = D_{[t,s']} + B_{[t,s']}(\mathbf{x}_t, q_t). \tag{35}$$

This is because, the LHS of (35) is the total depleted inventory during periods $t, t+1, \dots, s'$, which must equal to the total demand consumption and product outdated during those periods, i.e., the RHS of (35). Combining (35) and the continuity of $B_{[t,s']}(\mathbf{x}_t, q_t)$ in q_t, we have, when $q_t \ge K(\mathbf{x}_t)$,

$$B_{[t,s']}(\mathbf{x}_t, q_t) = B_{[t,s']}(\mathbf{x}_t, K(\mathbf{x}_t)) + q_t - K(\mathbf{x}_t).$$

Furthermore, since the inventory vector at the beginning of period $s'+1$ is $\mathbf{x}_{s'+1} = (u_{s'-m-L+2},\ldots,u_{s'})$ for any $q_t \geq K(\mathbf{x}_t)$, the products outdated during periods $s'+1,\ldots,s$ are independent of the order quantity q_t. Thus, when $q_t \geq K(\mathbf{x}_t)$, we have

$$B_{[t,s]}(\mathbf{x}_t,q_t) = B_{[t,s]}(\mathbf{x}_t,K(\mathbf{x}_t)) + q_t - K(\mathbf{x}_t). \tag{36}$$

Comparing (34) with (36), it remains to show that $B_{[t,s]}(\mathbf{x}_t,q_t) = B_{[t,s]}(\mathbf{x}_t,0)$ when $q_t < K(\mathbf{x}_t)$. This result directly follows from the following two observations: 1) the outdated products during periods $t,\ldots,t+m+L-2$ do not depend on q_t; and 2) by the definition of $K(\mathbf{x}_t)$, there is no product outdated during periods $t+m+L-1,\ldots,s$ when $q_t < K(\mathbf{x}_t)$. The proof is complete. **Q.E.D.**

Proof of Proposition 2. For clarity, we signify the dependence of $\widetilde{\Pi}_t^P(q_t)$ on $\mathbf{x}_t$ by using $\widetilde{\Pi}_t(\mathbf{x}_t,q_t)$. Then,

$$\widetilde{\Pi}_t^P(\mathbf{x}_t,q_t) = \sum_{s=t+L}^{T} (\Pi_s^{P(t)} - \Pi_s^{P(t-1)}),$$

where recall that $\Pi_s^{P(t)}$ is the backlogging cost incurred in period s under policy $P(t)$. Clearly, to prove the proposition, it suffices to show that it holds for $\Pi_s^{P(t)} - \Pi_s^{P(t-1)}$ for any $s \in \{t+L,\ldots,T\}$.

We first prove that $\Pi_s^{P(t)} - \Pi_s^{P(t-1)}$ is decreasing and convex in q_t for any $s \in \{t+L,\ldots,T\}$. Note that $\Pi_s^{P(t-1)}$ does not depend on q_t. Thus, we only need to prove that $\Pi_s^{P(t)}$ is decreasing and convex in q_t for any $s \in \{t+L,\ldots,T\}$. Since

$$\Pi_s^{P(t)} = \alpha^{s-1} b\Big(D_{[t,s]} + B_{[t,s)}(\mathbf{x}_t,q_t) - \Big(\sum_{i=1}^{m+L-1} x_{t,i} + q_t + \sum_{i=t+1}^{s-L} u_i\Big)\Big)^+,$$

the result is equivalent to proving that $B_{[t,s)}(\mathbf{x}_t,q_t) - q_t$ is decreasing and convex in q_t, which is true by Lemma 4.

We then prove that for any $s \in \{t+L,\ldots,T\}$, $\Pi_s^{P(t)} - \Pi_s^{P(t-1)}$ is decreasing in vector $\mathbf{x}_t$ and increasing in $(D_t,\ldots,D_s)$. First, since for any $t' \geq t+1$, $\mathbf{x}_{t'+1}$ is increasing in $\mathbf{x}_{t'}$ and decreasing in $D_{t'}$, it is easy to verify that both $\Pi_s^{P(t)}$ and $\Pi_s^{P(t-1)}$ are decreasing in $\mathbf{x}_t$ (as an increase in $\mathbf{x}_t$ can only reduce the shortage in period s) and increasing in $(D_t,\ldots,D_s)$ (since an increase in demand can only consume more inventory leading to a higher shortage cost in period s). Now, for any $(\mathbf{x}_t,q_t)$ in the region of $\mathbf{x}_t$ satisfying $\Pi_s^{P(t-1)} = 0$, we have $\Pi_s^{P(t)} - \Pi_s^{P(t-1)} = \Pi_s^{P(t)}$ is decreasing in $\mathbf{x}_t$ and increasing in $(D_t,\ldots,D_s)$.

Next, consider any state $\mathbf{x}_t$ in the region with $\Pi_s^{P(t-1)}(\mathbf{x}_t, q_t) > 0$. In this case, $\Pi_s^{P(t)}(\mathbf{x}_t, q_t) \geq \Pi_s^{P(t-1)}(\mathbf{x}_t, q_t) > 0$. Then, it follows from (8) that

$$\Pi_s^{P(t)} - \Pi_s^{P(t-1)} = \alpha^{s-1} b \left(B_{[t,s)}(\mathbf{x}_t, q_t) - B_{[t,s)}(\mathbf{x}_t, u_t) + \bar{q}_t \right).$$

Therefore, it is sufficient to prove that $B_{[t,s)}(\mathbf{x}_t, q_t) - B_{[t,s)}(\mathbf{x}_t, u_t)$ is decreasing in $\mathbf{x}_t$ and increasing in $(D_t, \ldots, D_s)$.

For $s \leq t + m + L - 1$, $B_{[t,s)}(\mathbf{x}_t, q_t)$ is independent of q_t by Lemma 4, thus the desired result is trivially true. For $s \geq t + m + L$, from Lemma 4, we have

$$B_{[t,s)}(\mathbf{x}_t, q_t) - B_{[t,s)}(\mathbf{x}_t, u_t) = (q_t - K(\mathbf{x}_t))^+ - (u_t - K(\mathbf{x}_t))^+,$$

which is increasing in $K(\mathbf{x}_t)$ since $q_t \leq u_t$. Hence, we only need to prove that $K(\mathbf{x}_t)$ is decreasing in $\mathbf{x}_t$ and increasing in $(D_t, \ldots, D_s)$. Note that $B_{[t,s]}(\mathbf{x}_t, q) - B_{[t,t+m+L-2]}(\mathbf{x}_t, q)$ is the total amount of outdated products in the interval $[t + m + L - 1, s]$ when the inventory vector is $\mathbf{x}_t$ at the beginning of period t and the system orders $q, u_{t+1}, \ldots, u_T$ in periods $t, t+1, \ldots, T$, respectively. Thus it can be seen that this quantity is increasing in $\mathbf{x}_t$ and decreasing in $(D_t, \ldots, D_s)$, and the desired property of $K(\mathbf{x}_t)$ follows. The proof of Proposition 2 is complete. **Q.E.D.**

C: Proof of Statements in §3.3

We first prove Proposition 3. Applying the definition of β in (10) yields

$$\gamma = 2 + \frac{\sum_{i=1}^{m+L-2} \alpha^i h}{\sum_{i=0}^{m-1} \alpha^{-i} h + \theta}. \tag{37}$$

Thus, $\gamma \leq 3$ is equivalent to $\theta/h \geq \Gamma(\alpha, m, L)$. Recall that θ and h are transformed cost parameters, given respectively by $\theta = \hat{\theta} + \alpha^{1-L}\hat{c} + \sum_{i=0}^{L-1} \alpha^{-i}\tilde{h}$ and $h = \hat{h} + \alpha^{-L}(1-\alpha)\hat{c} + (\alpha^{-L} - 1)\tilde{h}$. Substituting θ and h, we obtain that $\gamma \leq 3$ is equivalent to the inequality (12).

We next show that (14) is a sufficient condition for (12). Note that

$$\alpha^{1-L}(\alpha^{-m} + \alpha^{m+L-2} - 1) = (1 - \alpha^m)(\alpha^{1-m-L} - \alpha^{-1}) + \alpha^{-1} \geq \alpha^{-1}, \tag{38}$$

and from the definition of $\Gamma(\alpha, m, L)$ in (13), we have

$$\Gamma(\alpha, m, L) \leq \sum_{i=m+1}^{m+L-2} \alpha^i \leq \alpha^{m+1} \sum_{i=0}^{L-1} \alpha^i. \tag{39}$$

If the inequality (14) holds, then it follows that

$$\begin{aligned}\Gamma(\alpha,m,L)\hat{h} &\le \hat{\theta}+\Gamma(\alpha,m,L)\alpha^{-m-2}\tilde{h}+\alpha^{-1}\hat{c}\\ &\le \hat{\theta}+\alpha^{-1}(\sum_{i=0}^{L-1}\alpha^{i}\tilde{h}+\hat{c})\\ &\le \hat{\theta}+\alpha^{1-L}(\alpha^{-m}+\alpha^{m+L-2}-1)(\hat{c}+\sum_{i=0}^{L-1}\alpha^{i}\tilde{h}),\end{aligned}$$

where the second and third inequalities follow from the inequalities (39) and (38), respectively. Thus, (14) is a sufficient condition for (12).

We finally show that $\gamma \le 2+(m+L-2)\alpha/m$ and $\gamma \le 2+1/(\alpha^{-m}-1)$ when $\alpha<1$. From (37), we have

$$\gamma \le 2+\frac{\sum_{i=1}^{m+L-2}\alpha^{i}}{\sum_{i=0}^{m-1}\alpha^{-i}} \le 2+\frac{m+L-2}{m}\alpha.$$

In addition, when $\alpha<1$, we have

$$\gamma = 2+\frac{\sum_{i=1}^{m+L-2}\alpha^{i}}{\sum_{i=0}^{m-1}\alpha^{-i}} = 2+\frac{1-\alpha^{m+L-2}}{\alpha^{-m}-1} \le 2+\frac{1}{\alpha^{-m}-1}.$$

Q.E.D.

D: Proof of Proposition 5

We only need to prove that $\mathbf{1}(t\in\mathscr{T}_H \mid \mathscr{F}_t)$ is decreasing in $(D_t,\ldots,D_{t+L-1})$. If $Y_{t,t}^{PB}<Y_{t,t}^{OPT}$, then period t belongs to $\mathscr{T}_H$ with probability 1. In this case, $\mathbf{1}(t\in\mathscr{T}_H \mid \mathscr{F}_t)=1$, and the desired result obviously holds. In what follows, we prove that $\mathbf{1}(t\in\mathscr{T}_H \mid \mathscr{F}_t)$ is decreasing in $(D_t,\ldots,D_{t+L-1})$ when $Y_{t,t}^{PB}\ge Y_{t,t}^{OPT}$.

We define, for $t\le s<t+L$, the event

$$\mathscr{A}_{t,s} := [Y_{t,s}^{PB}\ge Y_{t,s}^{OPT}]\cap[Y_{t,s+1}^{PB}<Y_{t,s+1}^{OPT}].$$

Given $Y_{t,t}^{PB}\ge Y_{t,t}^{OPT}$, $t\in\mathscr{T}_H$ if and only if $\mathscr{A}_{t,s}$ occurs for some $s\in\{t,\ldots,t+L-1\}$. Suppose $\mathscr{A}_{t,s}$ occurs, i.e., $Y_{t,s}^{PB}\ge Y_{t,s}^{OPT}$, whereas $Y_{t,s+1}^{PB}<Y_{t,s+1}^{OPT}$. By (15) and since the two systems face the same demands, we must have $e_s^{PB}>0$. This implies that none of the products ordered after period $s-m-L+1$ has been consumed by demand at the beginning of period $s+1$, hence

$Y^{PB}_{t,s+1} = q^{PB}_{[s-m-L+2,t]}$. By the same argument, it can be seen that if $e^{PB}_s > 0$ and $q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1}$, then it holds that $Y^{PB}_{t,s+1} < Y^{OPT}_{t,s+1}$. This shows that we can rewrite $\mathscr{A}_{t,s}$ as

$$\mathscr{A}_{t,s} = [Y^{PB}_{t,s} \geq Y^{OPT}_{t,s}] \cap [e^{PB}_s > 0] \cap [q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1}]. \tag{40}$$

The observation above leads to the following important identity:

$$\begin{aligned} &\left\{\mathscr{F}_t, Y^{PB}_{t,t} \geq Y^{OPT}_{t,t}\right\} \cap [t \in \mathscr{T}_H] \\ &= \left\{\mathscr{F}_t, Y^{PB}_{t,t} \geq Y^{OPT}_{t,t}\right\} \bigcap \left\{ \bigcup_{s=t}^{t+L-1} [e^{PB}_s > 0] \bigcap \left[q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1}\right] \right\}. \end{aligned} \tag{41}$$

To see why this is true, first suppose the LHS occurs. Then, from the argument above, we know that $\mathscr{A}_{t,s}$ must occur for some period $s \in \{t, \dots, t+L-1\}$, and by (40), the RHS of (41) must occur too. Next, suppose the RHS of (41) occurs. Then, there must be some $s \in \{t, \dots, t+L-1\}$ satisfying $e^{PB}_s > 0$ and $q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1}$. As discussed above, $e^{PB}_s > 0$ implies $Y^{PB}_{t,s+1} = q^{PB}_{[s-m-L+2,t]}$, hence it further implies $Y^{PB}_{t,s+1} < Y^{OPT}_{t,s+1}$. By the definition of $\mathscr{T}_H$, this shows $t \in \mathscr{T}_H$ hence the LHS of (41) occurs too.

Recall that e^{PB}_s is the number of expired units in period s. These units are part of $q^{PB}_{s-m-L+1}$ ordered in period $s-m-L+1$ $(\leq t)$, and they are only affected by demands in period s or earlier. It is clear that, given $\mathscr{F}_t$, e^{PB}_s is decreasing in the demand process, hence $\mathbf{1}(e^{PB}_s > 0 \mid \mathscr{F}_t)$ is decreasing in $(D_t, D_{t+1}, \dots, D_{T-L})$. Similarly, $Y^{OPT}_{t,s+1}$ is the trimmed inventory position in OPT at period $s+1$ that is ordered at period t or earlier, which is also decreasing in $(D_t, D_{t+1}, \dots, D_{T-L})$. Since $q^{PB}_{[s-m-L+2,t]}$ is known at time t, we conclude that $\mathbf{1}(q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1} \mid \mathscr{F}_t)$ is also decreasing in $(D_t, D_{t+1}, \dots D_{T-L})$. By (41) and the assumption that $Y^{PB}_{t,t} \geq Y^{OPT}_{t,t}$ is implied by $\mathscr{F}_t$, we conclude that

$$\begin{aligned} \mathbf{1}\Big(t \in \mathscr{T}_H \mid \mathscr{F}_t\Big) &= \mathbf{1}\Bigg(\bigcup_{s=t}^{t+L-1} [e^{PB}_s > 0] \bigcap \left[q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1}\right] \,\Big|\, \mathscr{F}_t\Bigg) \\ &= 1 - \prod_{s=t}^{t+L-1} \left[1 - \mathbf{1}\Big(e^{PB}_s > 0 \mid \mathscr{F}_t\Big) \cdot \mathbf{1}\Big(q^{PB}_{[s-m-L+2,t]} < Y^{OPT}_{t,s+1} \mid \mathscr{F}_t\Big)\right] \end{aligned}$$

is decreasing in $(D_t, D_{t+1}, \dots, D_{T-L})$. This completes the proof of Proposition 5. **Q.E.D.**

E: Proof of Lemmas 1 - 3

Proof of Lemma 1. We first prove $\hat{q}^{PB}_t = 0$. First, if $\tilde{Y}^{PB}_t < 0$, then $Y^{PB}_t < 0$, i.e., the inventory position after placing the order q^{PB}_t in period t is negative. In this case, all the units in q^{PB}_t will meet the backlogged demand, and thus $\hat{q}^{PB}_t = 0$. Now suppose $\tilde{Y}^{PB}_t \geq 0$. In this case, since $t \in \mathscr{T}_{H1}$, we have $\tilde{Y}_t^{OPT} > \tilde{Y}_t^{PB} \geq 0$, and there exists an effective unit in $\tilde{Y}_t^{OPT}$, denoted by η^{OPT}, that cannot be paired to the units in $\tilde{Y}_t^{PB}$. Suppose $\hat{q}_t^{PB} > 0$. Then, there exists a perishing unit in $\hat{q}_t^{PB}$, denoted by η^{PB}. Since the unit η^{PB} is newly ordered in period t, it expires after the unit η^{OPT} meets the demand. By the FIFO issuance policy, the products ordered in the PB system after η^{PB} cannot be used to satisfy demand before η^{PB} expires. Thus, in the period when the demand unit for η^{OPT} arrives, the PB system will have both the perishing unit η^{PB} and backlogged demand, which results in a contradiction. This shows that $\hat{q}_t^{PB} = 0$.

We now analyze the holding cost of the effective units $\tilde{q}_t^{PB}$. Since $t \in \mathscr{T}_{H1}$, we have $\tilde{Y}_t^{OPT} > \tilde{Y}_t^{PB}$. Now focus on an arbitrary unit k in $\tilde{q}_t^{PB}$. Since k is an effective unit, its permanent pair k' of OPT exists. Because both systems face the same demand units and inventories are consumed by FIFO issuing policy, the unit k' must be ordered by OPT in or before period t, incurring no less holding cost than k. This completes the proof of the lemma. **Q.E.D.**

To prove Lemma 2, we first prove the key auxiliary Lemma 3.

Proof of Lemma 3. We first prove part (a). The perishing unit l is ordered in period t and perished in period $t+L+m-1$, incurring a discounted outdating cost $\alpha^{t+L+m-2}\theta$. Now we consider two cases.

Case 1: $n = 1$. In this case, since k_1 is ordered in or after period t, its total discounted holding cost is at most $\alpha^{t+L-1}\sum_{i=0}^{m-1}\alpha^i h$. The result in part (a) holds by noting that

$$\alpha^{t+L-1}\sum_{i=0}^{m-1}\alpha^i h \leq \frac{h}{\theta}\sum_{i=-m+1}^{m+L-2}\alpha^i \times \alpha^{t+L+m-2}\theta.$$

Case 2: $n \geq 2$. In this case, $k_1, \ldots, k_{n-1}$ are effective units of PB by our matching mechanism. We first argue, by contradiction, that the unit l is ordered before period $t_{l,1}$ (i.e., $t < t_{l,1}$). Suppose that this claim is not true. Then, l must be ordered in period $t_{l,1}$, the same period when k_1 is ordered. Since k_1 is an effective unit, there must be a demand unit that arrives before unit l expires and it is fulfilled by k_1 in PB. The corresponding demand unit in OPT is not fulfilled by the perishing unit l (while feasible); and by the FIFO issuance policy, it cannot be fulfilled by any unit ordered after period $t_{l,1}$ either. However, by our matching mechanism, the unit k_1', the permanent pair of k_1, fulfills this demand unit but it is ordered after period $t_{l,1}$, leading to a contradiction. This shows that unit l is ordered before period $t_{l,1}$.

For any $i = 1, \ldots, n-1$, by the mapping of l, the effective unit k_i is ordered by PB in period $t_{l,i}$, and by our matching mechanism, the permanent pair of k_i, i.e., k_i', is ordered by OPT before or in period $t_{l,i+1}$. Note that both units take L periods to enter the systems. Thus, the total discounted holding cost of unit k_i is at most that of unit k_i' plus $\sum_{i=t_{l,i}+L-1}^{t_{l,i+1}+L-2} h$. In addition, k_n is ordered in period $t_{l,n}$, and its total discounted holding cost is at most $\sum_{i=t_{l,n}+L-1}^{t_{l,n}+L+m-2} \alpha^i h$. Consequently, the total discounted holding cost of units $k_1,\dots,k_n$ is at most that of units $k'_1,\dots,k'_{n-1}$ plus $\sum_{i=t_{l,1}+L-1}^{t_{l,n}+L+m-2} \alpha^i h$. When $n \geq 2$, since $t < t_{l,1} < t_{l,n} \leq t+L+m-1$, we have

$$\sum_{i=t_{l,1}+L-1}^{t_{l,n}+L+m-2} \alpha^i h \leq \sum_{i=t+L}^{t+2L+2m-3} \alpha^i h \leq \frac{h}{\theta} \sum_{i=-m+1}^{m+L-2} \alpha^i \times \alpha^{t+L+m-2}\theta.$$

This shows that the desired result holds when $n \geq 2$, and completes the proof of part (a).

We next prove part (b). When $t \geq T-m-L+2$, the perishing unit l will still be in the system at the end of the planning horizon. By the FIFO issuance policy, all the effective units in OPT must have been ordered before or in period t. Thus, l will be paired with at most one perishing unit k_1 in PB. Since the mapping of l is nonempty, we must have $n=1$. Then, the result in part (b) follows from the fact that k_1 is ordered in or after period t. **Q.E.D.**

With Lemma 3 in place, we are now ready to prove Lemma 2.

Proof of Lemma 2. We divide $\sum_{t\in\mathscr{T}_{H2}} H_t^{PB}$, the total marginal holding cost of the units ordered by PB in periods $\mathscr{T}_{H2}$, into two parts: the total discounted holding cost of the units ordered in periods $\mathscr{T}_{H2}$ which are in the excess inventory of PB, and the total discounted holding cost of the *effective* units ordered in periods $\mathscr{T}_{H2}$ which are *not* in the excess inventory of PB. We have shown that any of the units ordered in periods $\mathscr{T}_{H2}$ which are in the excess inventory of PB *uniquely* belongs to the mapping of a perishing unit in OPT. In Lemma 3 we have shown that, for any perishing unit l with mapping $\{(t_{l,1},k_1),(t_{l,2},k_2),\dots,(t_{l,n},k_n)\}$, ordered by OPT before period $T-L-m+2$, the total discounted holding cost of the units $k_1,\dots,k_n$ in PB is at most that of the effective units $k'_1,\dots,k'_{n-1}$ plus $\sum_{i=-m+1}^{m+L-2} \alpha^i h/\theta$ times the discounted outdating cost of l; while for any perishing unit l ordered by OPT in or after period $T-L-m+2$, it is at most paired with one perishing unit k_1 in OPT, which incurs no more total discounted holding cost than l. After summing up all the perishing units in OPT, we obtain that the first part of cost in $\sum_{t\in\mathscr{T}_{H2}} H_t^{PB}$ is at most the sum of the following three terms:

1) The total discounted holding cost of the units in OPT that form permanent pairs with the *effective* units ordered in periods $\mathscr{T}_{H2}$ that are in the excess inventory of PB;

2) $\sum_{i=-m+1}^{m+L-2} \alpha^i h/\theta$ times the total discounted outdating cost of all outdating units in OPT, which equals $\sum_{i=-m+1}^{m+L-2} \alpha^i h \sum_{t=1}^{T-L} \Theta_t^{OPT}/\theta$;

3) The total discounted holding cost of the perishing units ordered in or after period $T-L-m+2$, which equals $\sum_{t=T-L-m+2}^{T-L} H_t^{OPT}(\hat{q}_t^{OPT})$.

Note that the second part of cost in $\sum_{t\in\mathscr{T}_{H2}} H_t^{PB}$ is at most the total discounted holding cost of the units in OPT that form permanent pairs with the *effective* units ordered in periods $\mathscr{T}_{H2}$ that are *not* in the excess inventory of PB. Combining these two statements together, we obtain

$$\sum_{t\in\mathscr{T}_{H2}} H_t^{PB} \le \sum_{t\in\mathscr{T}_{H2}} \hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB}) + \sum_{t=T-L-m+2}^{T-L} H_t^{OPT}(\hat{q}_t^{OPT}) + \frac{h}{\theta}\sum_{i=-m+1}^{m+L-2}\alpha^i \sum_{t=1}^{T-L}\Theta_t^{OPT}$$
$$= \sum_{t\in\mathscr{T}_{H2}} \hat{\mathbf{H}}^{OPT}(\tilde{q}_t^{PB}) + \sum_{t=1}^{T-L} H_t^{OPT}(\hat{q}_t^{OPT}) + \frac{h}{\theta}\sum_{i=1}^{m+L-2}\alpha^i \sum_{t=1}^{T-L}\Theta_t^{OPT},$$

where the equality follows from

$$\sum_{t=1}^{T-L-m+1} H_t^{OPT}(\hat{q}_t^{OPT}) = \frac{h}{\theta}\sum_{i=-m+1}^{0}\alpha^i \sum_{t=1}^{T-L}\Theta_t$$

as each perishing unit ordered in period t, with $t < T-L-m+2$, incurs an outdating cost $\alpha^{t+L+m-2}\theta$ and a total discounted holding cost $\alpha^{t+L+m-2}\sum_{i=-m+1}^{0}\alpha^i h$. **Q.E.D.**

F: Proof of Proposition 7

For convenience, we let $W_{ts}^P = (\Pi_s^{P(t)} - \Pi_s^{P(t-1)})/(\alpha^{s-1}b)$ denote the shortage in period s resulting from decision q_t made in period t that could have been avoided by ordering the full capacity.

By the identities

$$\sum_{t\in\mathscr{T}_\Pi} \widetilde{\Pi}_t^{PB} = \sum_{t\in\mathscr{T}_\Pi}\sum_{s=t+L}^{T}\alpha^{s-1}bW_{ts}^{PB} = \sum_{s=1+L}^{T}\sum_{1\le t\le s-L, t\in\mathscr{T}_\Pi}\alpha^{s-1}bW_{ts}^{PB},$$
$$\sum_{t=1}^{T-L} \widetilde{\Pi}_t^{OPT} = \sum_{t=1}^{T-L}\sum_{s=t+L}^{T}\alpha^{s-1}bW_{ts}^{OPT} = \sum_{s=1+L}^{T}\sum_{t=1}^{s-L}\alpha^{s-1}bW_{ts}^{OPT},$$

it is sufficient to prove that, for any period $s = L+1,\ldots,T$,

$$\sum_{t=1}^{s-L} W_{ts}^{OPT} \ge \sum_{1\le t\le s-L, t\in\mathscr{T}_\Pi} W_{ts}^{PB}. \tag{42}$$

Fix a period s. If the set $\{t : 1\le t\le s-L, t\in\mathscr{T}_\Pi\}$ is empty, then (42) is automatically satisfied. Otherwise, let s' denote the largest element in the set. Then clearly we have

$$\sum_{t=1}^{s'} W_{ts}^{PB} \ge \sum_{1\le t\le s-L, t\in\mathscr{T}_\Pi} W_{ts}^{PB}, \quad \text{and} \quad \sum_{t=1}^{s'} W_{ts}^{OPT} \le \sum_{t=1}^{s-L} W_{ts}^{OPT}.$$

Hence, to establish (42), it suffices to prove that

$$\Pi_s^{PB(s')} = \sum_{t=1}^{s'} \alpha^{s-1} b W_{ts}^{PB} + \Pi_s^{P(0)} \leq \sum_{t=1}^{s'} \alpha^{s-1} b W_{ts}^{OPT} + \Pi_s^{P(0)} = \Pi_s^{OPT(s')}, \tag{43}$$

where the two equalities follow from the definition of W_{ts}^P. In what follows, we prove the inequality in (43) by considering two different cases.

Case 1: $s \leq s'+L+m-1$. In this case, it follows from the definition of $Y_{s',s}$ that $Y_{s',s}^{PB} + u_{(s',s-L]}$ is the total on-hand inventory level at the beginning of period s under policy $PB(s')$. Thus,

$$\Pi_s^{PB(s')} = \alpha^{s-1} b (D_s - Y_{s',s}^{PB} - u_{(s',s-L]})^+.$$

Similarly, $\Pi_s^{OPT(s')} = \alpha^{s-1} b (D_s - Y_{s',s}^{OPT} - u_{(s',s-L]})^+$. Since $s' \in \mathscr{T}_\Pi$, we have $Y_{s',s}^{PB} \geq Y_{s,s}^{OPT}$. Thus, $\Pi_s^{PB(s')} \leq \Pi_s^{OPT(s')}$, establishing (43) under Case 1.

Case 2: $s > s'+L+m-1$. In this case, since $s' \in \mathscr{T}_\Pi$, we have $Y_{s',s'+L+m-1}^{PB} \geq Y_{s',s'+L+m-1}^{OPT}$. Note that $Y_{s',s'+L+m-1}^{PB}$ is the part of the inventory position at the beginning of period $s'+L+m-1$ that is ordered in period s' or earlier, and the products ordered before period s' would have been either consumed or expired before period $s'+L+m-1$. When $Y_{s',s'+L+m-1}^{PB} \geq 0$, $Y_{s',s'+L+m-1}^{PB}$ is the on-hand inventory level of products under policy PB which were ordered in period s', and when $Y_{s',s'+L+m-1}^{PB} < 0$, $-Y_{s',s'+L+m-1}^{PB}$ represents the amount of "*backlogged*" demand at the beginning of period $s'+m+L-1$ that needs to be fulfilled by products ordered after period s' (in an FIFO manner). Since $Y_{s',s'+L+m-1}^{PB} \geq Y_{s',s'+L+m-1}^{OPT}$ and the policies $PB(s')$ and $OPT(s')$ order the same full capacity in every period after period s', one can verify that the inventory vector $\mathbf{x}_{s'+L+m-1}^{PB(s')}$ *weakly* dominates the inventory vector $\mathbf{x}_{s'+L+m-1}^{OPT(s')}$ in every component. Furthermore, for each realization f_T, it is easy to show, by induction and using system dynamics, that $\mathbf{x}_t^{PB(s')}$ weakly dominates $\mathbf{x}_t^{OPT(s')}$ in every component for any $t \geq s'+L+m-1$, and in particular for period s. Thus, it follows that the regular backlogging cost $\Pi_s^{PB(s')}$ in period s is no greater than $\Pi_s^{OPT(s')}$, establishing (43) under Case 2.

Since we have shown that (43) holds for an arbitrary s, the proof is complete. **Q.E.D.**